This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

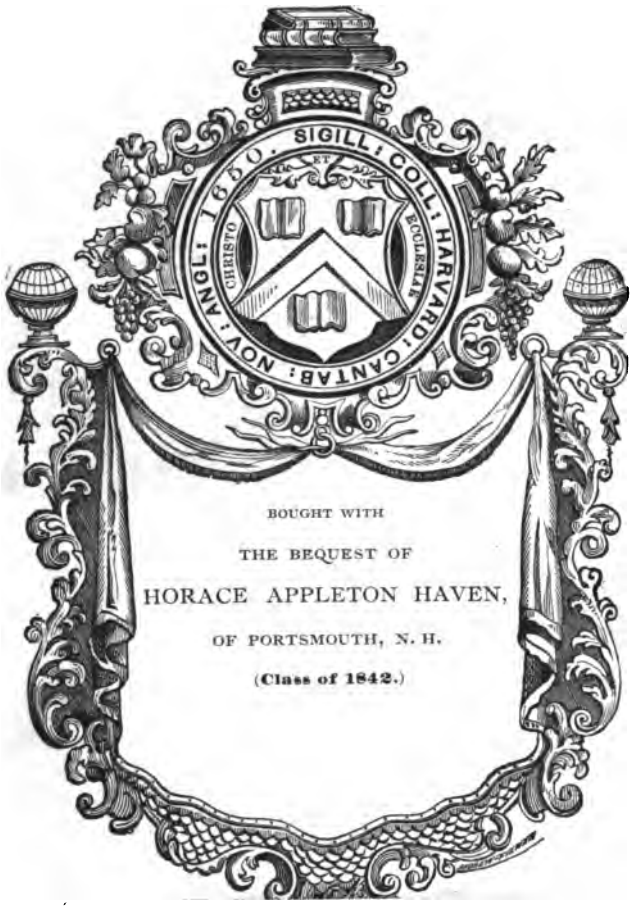
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Math 388.86



SCIENCE CENTER LIBRARY

MATHEMATICAL

QUESTIONS AND SOLUTIONS

FROM "THE EDUCATIONAL TIMES."

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN "THE EDUCATIONAL TIMES."

EDITED BY

CONSTANCE I. MARKS, B.A.

NEW SERIES.

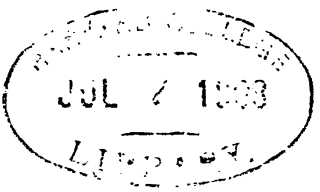
VOLUME XIII.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1908.

Math 388.86
(CXIII. 29)



Haven Fund
(XIII)

CONTRIBUTORS.

ABBOTT, R. C., B.A.; Marlborough College.
ADAIR, E. W.; Guernsey.
ALEXANDER, J. J., M.A.; Tavistock.
ALLEN, Rev. A. J. C., M.A.; Cambridge.
ANDERSON, Prof., M.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; Westwood, Hereford.
ANTONIO, J.; Penang.
APPLETON, Miss E.; Liverpool.
ARCHIBALD, R. C., M.A., Ph.D.; Acadia Univ., Nova Scotia, Canada.
ARMENANTE, Professor; Pesaro.
ARNOLD, I.; Belfast.
ARUNACHALAN, M. V., M.A.; The Nizam Coll., Hyderabad.
AYRTON, Mrs.; Hyde Park, W.
BAGCHI, S. C., B.A.; Cambridge.
BALL, Sir ROBERT STAWELL, LL.D., F.R.S., Professor of Astronomy and Geometry, Camb.
BALL, W. W. ROUSE; Cambridge.
BARNIVILLE, JOHN J., B.A.; I.C.S., Chittagong, Bengal.
BARRETT, E. P., B.A.; Stroud.
BARTON, W. J., M.A.; Highgate, London.
BATEMAN, H. B.A.; Cambridge.
BATES, G.N., M.A.; Ramsgate.
BATTAGLINI, Prof. GIUSEPPE; Univ. di Roma.
BEARD, W. F., M.A., Melbourne.
BELL, A. H.; Litchfield, Ill., U.S.A.
BEMAN, Prof. W. W.; Univ. Michigan.
BESANT, W. H., D.Sc., F.R.S.; Cambridge.
BEYENS, Prof. IGNACIO, Lt.-Col. du Génie; Cadiz.
BHATTACHARYYA, Prof. N.
BHUT, Professor ATH BIJAH, M.A.; Dacca.
BICKERDIKE, C.; Scarborough.
BIDDLE, D. (Editor, Vols. LXVII.–LXXV., Original Series); Kingston-on-Thames.
BIGGS, R., M.A., LL.D.; Portora, Enniskillen.
BILLUPS, H. B.; Sandown, I. of W.
BIRCH, Rev. J. G., M.A.; Bunralty, co. Clare.
BIRTWISTLE, G., B.A., B.Sc.; Pemb. Coll., Camb.
BLAIKIE, J., M.A.; Norwood.
BLATER, JOSEPH; Baden Baden.
BLYTHE, W. H., M.A.; St. Mark's Lodge, Newn-
BOLAND, J.; Sand Hill, Gibraltar. [ham.
BOOTH, Prof. W.; Hoogly College, Bengal.
BORDAGE, Prof. EDMOND; Coll. de Nantua.
BOURNE, Rev. C. W., M.A.; King's Coll. Sch.
BOUTON, V. J., B.Sc.; Hampton Wick.
BOWSER, Prof. E. A.; New Jersey, U.S.A.
BOYS, Prof.; Royal College of Science.
BRILL, J., M.A.; Cambridge.
BROCARD, H., Chef de Bataillon; Bar-le-Duc.
BROOKS, Prof. E.; Millersville, Pennsylvania.
BROWN, Prof.; Providence, U.S.A.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof. JOSEPH; Dublin.
BRUCE, Rev. ROBERT, D.D.; Huddersfield.
BRUNYATE, W. E.; Trinity Coll., Cambridge.
BRYAN, G. H., F.R.S.; Bangor.
BRYANT, R., D.Sc., B.A.; London Institution.
BUCK, EDWARD, M.A.; Oakhill, Bath.
BURKE, J., M.A.; Trinity College, Dublin.
BURNSIDE, Professor, M.A.; Univ. of Dublin.
BURSTALL, H. J. W.; King's Coll., London.
CAMA, B. N., M.A.; St. John's Coll., Cambridge.
CAMPBELL, J. E., M.A.; Hertford Coll., Oxford.
CANHAM, D.; Norwich.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., M.A.; Chavey Down, Bracknell, Berks.
CATALÁ, Prof. L.; Cadiz.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAVE, F. E.; Girton Coll., Cambridge.
CHAKRAVARTI, Professor BYOMAKESA, M.A. Calcutta.
CHANDRA BASÛ, Professor, M.A.; Calcutta.
CHAPMAN, Prof. R. W., M.A.; Univ. of Adelaide.
CHARTRES, R.; 72 Carter Street, Manchester.
CHRISTIE, R. W. D.; Stanley Coll., Liverpool.
CLARK, A.; Aberystwyth.
CLARKE, Colonel A. R., C.B., F.R.S.; Redhill.
CLAYTON, Prof., M.A.; Parkville, Melbourne.
CLIFFORD, Prof. W. K., F.R.S.
COCHEZ, Professor; Paris.
COHEN, ARTHUR, M.A., K.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrews.
CONSTABLE, Rev. W. J., M.A.; Uppingham.
COONDOO, Prof. MAHENDRA NATH; Benares.
COONEY, W. S.; High School, Dublin.
COOPER, T. M. A., M.A.; Cheltenham.
COTTERILL, J. H., M.A.; St. Alban's Mansions, W.
CRAWFORD, G. E., M.A.; Clifton, Bristol.
CREMONA, Prof. LUIGI; Rome.
CROFTON, Professor M. W., F.R.S.; Ryde.
CROKE, J. O'BYRNE, M.A.; Dublin.
CULLEN, Rev. J.; Mount St. Mary's Coll., Chesterfield.
CULLEY, Prof., M.A.; St. David's Coll., Lampeter.
CUNNINGHAM, ALLAN, Lt.-Col., R.E.; Kensington
CURJEL, H. W., M.A.; Southport.
CURTIS, Prof., M.A., S.J.; Univ. Coll., Dublin.
CZUBER, Dr. EMANUEL; Prag, Bohemia.
DALLAS, R. J.; Pemberton Gardens, N.
DANIEL, VINCENT, B.Sc.
DANIELS, A. E., B.A.; Nottingham.
DARBOUX, Professor; Paris.
DATA, Prof. PROMATHANATH, M.A.; Delhi.
DAVIES, Prof., Brown Univ., Providence, U.S.A.
DAVIS, Rev. B., M.A.; Principal of Jai Narayan College, Benares.
DAVIS, R. F., M.A.; Acton Green.
DAWSON, H. G., M.A.; Sheffield.
DE LONGCHAMPS, Professor; Paris,
DE VRIES, Professor JAN; Utrecht.
DE WACHTER, Professor, M.A.; Schaarbeck.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DIBB, J. H., B.Sc.; Hendon.
DICK, Hon. G. R., M.A.; Auditor Gen., Port Louis, Mauritius.
DICKSON, J. D. H., M.A.; Peterhouse, Camb.
DICKSON, W.; Edinburgh.
DOBBS, W. J., M.A.; E. Putney.
D'OCAGNE, MAURICE; Paris.
DROZ-FARNY, Prof. A.; Collège de Porrentruy.
DRURY, H. D., M.A.; Marlborough College.
DUPAIN, J. C.; Professor au Lycée d'Angoulême.
DURELL, Rev. C. V., M.A.; Fulbourn Rectory near Cambridge.
EASTON, BELLE, B.Sc.; Lockport, New York.
EDMONDSON, Prof. T. W.; New York Univ.
EDWARDES, DAVID, B.A.; Bangor.
ELLIOTT, Prof. E. B., F.R.S.; F. Mag. Coll., Oxon.; Oxford.
ELTON, E. F., M.A.; Wellington College.
EMMERICH, Prof., Ph.D.; Mülheim-am-Ruhr.
ESCOTT, Prof. E. B.; Univ. of Michigan, U.S.A.
EVE, A. E., M.A.; Marlborough College.
EVERETT, Prof. J. D., D.C.L.
EWIN, CECIL; Drumcondra, Dublin.
FINKEL, B. F.; Missouri, U.S.A.

FLOOD, P. W.; Drumcondra, Dublin.
FORSYTH, Prof., Sc.D.; Cambridge.
FORTEY, H., M.A.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.
FOSTER, W. S.; Hoddesdon, Herts.
FRANKLIN, Prof. CHRISTINE LADD, M.A., B.Sc.
FREETH, Rev. Dr.
GALLIERS, Rev. T., M.A.; Kirstead Rect., Norwich
GANGULI, SARADAKANTA, M.A.; Calcutta.
GASKIN, C. H.; Birkenhead.
GENESE, Prof., M.A.; Univ. Coll., Aberystwyth.
GEORGI, MAURICE; Paris.
GERRANS, H.T., M.A.; Fel. & Tut. Wor. Coll., Oxon.
GHOSH, SRISH CHANDRA, M.A.; Calcutta.
GHOSH, Prof. U. CHANDRA; Allahabad.
GIBSON, T. PAUL; St. Cloud, France.
GLAISHER, J. W. L., F.R.S.; Fel. Trin. Coll., Camb.
GOLDENBERG, Professor, M.A.; Moscow.
GOPALACHARI, Prof.; Presidency Coll., Madras.
GORDON, A., M.A.; Barnwood Ho., Gloucester.
GOULD, S. C.; New Hampshire, U.S.A.
GRACE, G. M., B.A., B.Sc.; Hammersmith.
GRACE, J. H.; Peterhouse, Cambridge.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GRAHAM, THOMAS; Kilkeel, Co. Down.
GREENHILL, A.G., M.A., F.R.S.; Artil.C., W'wich.
GREENSTREET, W. J., M.A.; Marling Sch., Stroud.
GREENWOOD, JAMES M.; Kansas City.
GRIFFITH, GERTRUDE, B.A.; Southampton.
GRIFFITHS, D. T., M.A.; Caermarthen.
GRIFFITHS, G. J., M.A.; Fellow Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROSS, W., LL.D.; Bournemouth.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
GUNNING, S., J.P.; Cookstown, Co. Tyrone.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMILTON, J. P.; Penang.
HAMMOND, J., M.A.; 14 Granville Square, W.C.
HANUMANTA RAU, Professor, B.A.; Madras.
HARDY, G. H., M.A.; Trinity Coll., Cambridge.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Rev. R., M A., F.R.S.; Forest Hill.
HATTON, J. L. S.; E. Lond. Tech. Coll.
HAY, THEODORE W.; Leven, Fife.
HEAL, W. E., M.A.; Marion, Ind., U.S.A.
HENDERSON, R., B.A., Queen's Coll., Belfast.
HEPPEL, GEORGE, M.A.; Ealing.
HERVEY, F. R. J., M.A.; Worthing.
HILL, Rev. E., M.A.; Cockfield, Suffolk.
HILL, F. W.; City of London School.
HILL, Prof. M. J. M., M.A., D.Sc., F.R.S.; Univ. Coll. Lond.
HILLHOUSE, Dr. W.; Newhaven, Conn., U.S.A.
HILLS, G. F. S.; Trin. Coll., Camb.
HILLYER, C. E., M.A.; Cambridge.
HIME, Lt.-Col.; Alexandria.
HINTON, C. H., M.A.; Cheltenham College.
HIPPISLEY, R. L., Major, R.E.; Farnborough.
HOBSON, Dr.; Cambridge.
HODGKINSON, J.; Oxford.
HOLM, ALEXANDER, M.A.; Glasgow.
HOLT, J. R., M.A.; Dublin.
HOOKER, J. H., M.A.; 184 Evering Road, N.E.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOROBIN, C. J., B.A.; M.L.S.C.
HOWSE, G. F.; Balliol College, Oxford.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, J. F.; Gunnersbury House, Acton.
HUDSON, Prof. W. H. H., M.A.; Croydon.
HUGHES, EDITH K. A.; Stroud.
HUGHES, W. R; Jesus College, Cambridge.
HUNTINGTON, E.V., Ph.D.; Camb., Mass., U.S.A.
IKIN, A. E., B.Sc.; Kettering.
ISSERLIS, L., B.A.; Munic. Tech. Inst., West Ham.
JACKSON, Rev. F. H., R.N., M.A.; China Station, Malta.
JACKSON, Miss F. H.; Towson, Baltimore.
JACKSON, W. H., B.A.; Clare Coll., Camb.
JACOBS, H. J.; London Institution, Finsbury.
JAGO, G.; Plymouth Public School.
JENKINS, MORGAN, M.A.; Tunbridge Wells.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, J. A. H., M.A.; Tonbridge.
JOHNSTONE, W.J., M.A.; Univ. Coll., Aberystwyth
JOLLIFFE, A. E., M.A.; St. David's College.
JOURDAIN, P. E. B.; Broadwindsor, Dorset.
KAHN, A., B A.; Mildmay Grove, N.
KALIPADU BASO, Prof., M.A.; Dacca College.
KATHERN, R. P., M.A.; Fleetwood, Bath.
KELVIN, Lord, F.R.S.; Largs, Ayrshire.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KITCHIN, F. L., B.A., R.N.; Exmouth.
KITCHIN, Rev. J. L., M.A.; Exmouth.
KITTUDGE, LIZZIE A.; Boston, United States.
KLEIN, Professor FELIX; Univ. Göttingen.
KNISELEY, ALEX.; Columbia City, Indiana.
KNOWLES, R., L.C.P.; Tottenham.
KOEHLER, J.; Rue St. Jacques, Paris.
KOLBE, Rev. Dr.; St. Mary's Coll., Cape Town.
KRISHMACHANDRA DE, Prof.; Pudukkottai.
KRISHNAMACHARRY, Prof., M.A.; Tiupati, India.
LACHLAN, R., M.A.; Newnham Terrace, Camb.
LAISANT, C. A.; Paris.
LAMB, R.; George Town, Penang.
LAMPE, Prof., Ed. of *Jahrb. der Math.*; Berlin.
LANGLEY, E. M., B.A.; Adelaide Sq., Bedford.
LASKER, E., Ph.D.; Manchester.
LAUVERNAY, Prof.
LAVERTY, W.H., M.A.; late Exam. in Univ. Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LAWSON, J. S., M.A.; Newcastle-on-Tyne.
LAYNG, A. E.; Grammar School, Stafford.
LEGABEKE, Professor; Delft.
LEIDHOLD, R., M.A.; Finsbury Park.
LEMOINE, E.; Paris.
LEUDESDORF, C., M.A.; Fel. Pembroke C., Oxon.
LEWIS, JOHN; Cardiff.
LEWIS, WM., M.A.; Headmaster, Llandyssul.
LIAGRE, General; Rue Carotz, Brussels.
LONDON, Rev. H., M.A.; Wimbledon.
LONGHURST, Rev. E. S., B.A.; Newark.
LOWRY, Rev. W. H., M.A.; Manitoba.
LOXTON, C. A.; Cannock.
LUDWIG, Professor; North Carolina, U.S.A.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.
MACAULAY, F. S., M.A.; St. Paul's School.
MCCLELLAND, W.J., B.A.; Prin. of Santry School.
MACCOLL, HUGH, B.A.; Boulogne.
MCCOY, Rev. R.; Stonyhurst.
MCCUBBIN, J., B.A.; Burgh Academy, Kilsyth.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, Prof. A., D.Sc.; Univ. of Texas.
MCINTOSH, ALEX., B.A.; Bedford Row, London.
MACKAY, J. S.; Edinburgh.
MACLEOD, J., H.M.I.S., I.S.O.; Elgin.
MACMAHON, Major P. A., F.R.S.; R.M.Academy.
MCMAHON, Prof., M.A.; Cornell Univ., Ithaca.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MCVICKER, C. E., M.A.; Kensington.
MADDISON, ISABEL, B.A.; Brynmawr, Pa., U.S.A.
MADHAVARAO, Prof., M.A.; Vizianagram.
MAINPRISE, B. W.; R. Naval School, Eltham.
MALET, J. C., M.A., F.R.S.; Dublin.
MANN, M. J. J.; Cromwell Road, London.
MANNHEIM, M.; Prof. à l'École Polytech. Paris.
MARKS, Miss CONSTANCE I., B.A. (Editor); W. Kensington.
MARTIN, ARTEMAS, M.A., Ph.D.; Washington.
MASSEY, W. H.; Twyford, Berks.
MATHEWS, Professor G. B., M.A.; St. John's Coll., Cambridge.
MATZ, Prof., M.A.; King's Mountain, Carolina.
MAYERS, F. N., M.A.; Argyle Square, W.C.
MELLOR, J. W., B.Sc.; Owens Coll., M'chester.
MERRIFIELD J., LL.D., F.R.A.S.; Plymouth.
MERRIMAN, MANSFIELD, M.A.; Yale College.
METCALFE, G., M.A.; Wimbledon.
MEYER, MARY S.; Girton College, Cambridge.
MILLER, W. J. C., B.A. (Editor, Vols. I.–LXVI., Original Series).
MILNE, Rev. J. J., M.A.
MINCHIN, G. M., M.A.; Prof. in Coopers Hill Coll.
MITCHELL, HUGH, M.A.; Edinburgh.

MITCHESON, Rev. T., B.A.
MITTAG-LEFFLER, Professor; Stockholm.
MONCK, H. ST., M.A.; Trin. Coll., Dublin.
MOORE, H. K., B.A.; Trin. Coll., Dublin.
MOREL, Professor; Paris.
MORGAN, C., M.A.; R. Naval Coll., Greenwich.
MORLEY, Prof., Sc.D.; Johns Hopkins Univ., Baltimore.
MORRICE, G. G., M.A., M.D.; Weymouth.
MUGGERIDGE, G. D., B.A.; Pemb. Coll., Camb.
MUIR, THOMAS, M.A., F.R.S.; Cape Town.
MUIRHEAD, R. F.; Glasgow.
MUKHOPADHYAY, Prof. ASUTOSH, M.A., F.R.S.E.
MUKHOPADHYAY. Prof. SYAMADAS; Chinsurah.
MULCASTER, J. W., M.A.; Allendale Town.
MUNINDRA, NATH RAY, Prof. M.A., LL.B.
NANSON, Prof. E. J.; Melbourne University.
NEALE, C. M.; Middle Temple, E.C.
NESBITT, A. M., M.A.; Melbourne.
NESBITT, H. A.; Hampstead.
NEUBERG, Professor; University of Liége.
NEWCOMB, Prof. SIMON, M.A.; Washington.
NISNER, I.; Westbourne Grove, W.
NIXON, R. C. J., M.A.
NORTH, FRANCIS C.; Ashton-on-Mersey.
O'CONNELL, Major-General P.; Cheltenham.
OLDHAM, C. H., B.A., B.L.; Dublin.
OPENSHAW, Rev. T. W., M.A.; Clifton, Bristol.
ORCHARD, JOHN, M.A.
ORCHARD, Professor, M.A., B.Sc.; Hampstead.
O'REGAN, JOHN; Limerick.
ORPEUR, HERBERT; East Dulwich.
OWEN, J. A., B.Sc.; Tennyson St., Liverpool.
PANCHAPAGESA AIYAR, K.G.; Madras.
PANTON, A.W., M.A.; Fell. of Trin. Coll., Dublin.
PARANJPYE, R. P., B.A.; Poona, India.
PAROLING, Professor; Cornell University.
PEACHELL, F. H., M.A.; Weymouth.
PEELE, T., A.I.A.; Manchester.
PEIRCE, Prof. B. O.; Harvard College.
PEIRIS, M. R.; Moratuwa, Ceylon.
PERRIN, EMILY, B.Sc.; St. John's Wood Park.
PHILLIPS, F. B. W.; 7 Harpur Place, Bedford.
PILLAI, Professeur, M.A.; Trichy., Madras.
PLAMENEWSKI, H., M.A.; Daghestan.
POCKLINGTON, H. C., M.A.; Yorks Coll., Leeds.
POLIGNAC, Prince CAMILLE DE.
POLLEXFEN, H., B.A.; Windermere College.
POOLE, A. W.; London.
POOLE, GERTRUDE, B.A.; Cheltenham.
PORTELA, Señor FRANCISCO DE V.; Cadiz.
PORTER, FRANK, M.A.
PRESCOTT, J., B.A.; Manningham Lane, Bradf'd.
PRESSLAND, A. J., M.A.; Academy, Edinburgh.
PRESTON, G. W., B.A.; Wheatley, near Halifax.
PRICE, H., M.A.; Penarth.
PRUDDEN, FRANCES E.; Lockport, New York.
PUCKLE, G. HALE, M.A.; Windermere.
PURSER, FREDK., M.A.; Fell. Trin. Coll., Dublin.
PURSER, Prof., M.A.; Queen's Coll., Belfast.
PUTNAM, K. S., M.A.; Rome, New York.
PYDDOKE, H. W., M.A.; Oxford.
QUIN, A. G. B.; Streatham.
QUINT, Dr.; Gravenhäge, Netherlands.
RADHAKRISHNAN, Prof., M.A.; Pudukkottai.
RAMA AIYANGAR, Prof.; Trichinopoly, Madras.
RAMACHANDRA ROW, Prof., M.A.; Trichinopoly.
RAMSEY, A. S., M.A.; Fettes Coll., Edinburgh.
RANGACHARI, V. SRINIVASA, B.A.; Government College, Kumbakonum, India.
RAO, K. DEVA, B.A.; Mangalore.
RAU, ALBERT G., B.S.; Bethlehem, Pa., U.S.A.
RAU, V. RAGUNATH; Torupati, India.
RAWSON, ROBERT; Havant, Hants.
REAY, LIONEL E., B.A.; Canterbury.
REES, E. W.; Penarth, Cardiff.
REEVES, F. W., B.A.
REEVES, G. M., M.A.; Lee, Kent.
REYNOLDS, B., M.A.; Cambridge.
RICE, JAMES; Craigie Road, Belfast.
RICHARDS, DAVID, B.A.; Christ's Coll., Brecon.
RICHARDSON, A. R.
RICHARDSON, Rev. G., M.A.; Winchester Coll.
RIDDELL, H., M.E.; Belfast.
ROACH, Rev. T., M.A.; Twyford, Hants.
ROBB, A. A.; St. John's Coll., Cambridge.
ROBERTS, A. D., B.A.; Jesus Coll., Oxford.
ROBERTS, R. A., M.A.; New York.
ROBERTS, W. R. W., M.A.; Fell. of Trin. Coll., Dub.
ROBERTSON, T. BRAILSFORD; S. Australia.
ROBSON, H. C., B.A.; Sidney Sussex Coll., Camb.
ROGEL, Professor; Mittweida, Germany.
ROGERS, L. J., M.A.; Yorkshire Coll., Leeds.
ROSS, C. M.; King's Coll., Cambridge.
ROWTON, Rev. S. J., M.A., Mus. Doc.; Folkestone.
ROY, Professor KALIPRASANNA, M.A.; Agra.
RUGGERO, SIMONELLI; Università di Roma.
RUSSELL, ALEXANDER, B.A.; Oxford.
RUSSELL, B.; Pembroke Lodge, Richmond.
RUSSELL, J. W., M.A.; Merton College, Oxford.
RUSSELL, R., M.A.; Fell. of Trin. Coll., Dublin.
RUSSELL, T.; Caius College, Cambridge.
RUTTER, E.; Sunderland.
SADLER, JOHN; Birkenhead.
ST. CLAIR, J. C., M.A.; Ryder Street, London.
SALMON, Rev. GEORGE, D.D., F.R.S.
SALMON, W. H., B.A., B.Sc.; Bedford.
SANDBERG, C. F., M.A.; Oxford.
SANDERSON, Rev. C. M., M.A.; Brington Rectory.
SANJANA, Prof., M.A.; Bhavnagar, Bombay.
SARADARANJAN RAY, Prof., M.A.; Dacca.
SARKAR, Prof. BENI MADHAV, M.A.; Agra.
SARKAR, Prof. NILKANTHA, M.A.; Calcutta.
SASTRY, M. V. A., B.A.; Saidapet, Madras.
SAVAGE, Prof. T.
SCHARAR, C. H.; Scranton, U.S.A.
SCHEFFER, Professor; Mercersbury Coll., Pa.
SCHOUTE, Prof. P. H.; University of Groningen.
SCOTT, A.W., M.A.; St. David's Coll., Lampeter.
SCOTT, Professor CHARLOTTE A., D.Sc.
SCOTT, GEORGE, M.A.; Greystones.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Camb.
SCRIMGEOUR, W., M.A., B.Sc.; Peebles.
SEARLE, C.; Clapham Common.
SEGAR, HUGH W.; Univ. Coll., Auckland.
SEN, Prof. RAJ. MOHAN; RajshabyeColl., Bengal.
SEWELL, Rev. H., B.A.; Bury Grammar School.
SEYMOUR, W. R., M.A.; St. Leonards.
SHARPE, J. W., M.A.; Bournemouth.
SHEPHERD, Rev. A. J. P., B.A.; Fellow of Queen's College, Oxford.
SHIELDS, Prof., M.A.; Coopwood, Mississippi.
SIMMONS, Rev. T. C., M.A.; Grainthorpe Vicarage, North Thoresby S. O., Lincs.
SIRCOM, Prof. SEBASTIAN, M.A.; Liverpool.
SITARAMAIYAR, Prof., Hindu Coll., Tinnevelly.
SIVERLEY, WALTER; Oil City, Pennsylvania.
SKRIMSHIRE, Rev. E., M.A.; Llandaff.
SLEIGH, W. A., B.A.; The Mercers' Sch., Holborn
SMITH, C., M.A.; Sidney Sussex College, Camb.
SMITH, Professor D. E.; Brockport, N.Y.
SMITH, J. G.; Portora, Enniskillen.
SMYLY, J. G.; Merrion Square, Dublin.
SOMMERVILLE, D.M.Y., M.A., D.Sc.; St. Andrews.
SOPER, H. E., B.A.; Highgate.
SPARK, Rev. F. M.; Lewisham, London.
STANHAM, W. C., B.A.; Kensington.
STEEDE, B.H., M.A.; Trinity College, Dublin.
STEGGALL, Prof. J. E. A., M.A.; Dundee.
STEPHEN, ST. JOHN, B.A.; Caius Coll., Cambridge.
STEWART, H., M.A.; Islington.
STOOPS, J. M., B.A.; Victoria College, Belfast.
STOOPS, W.; Newry.
STORMONTH, JAMES; Edinburgh.
STORR, G. G., M.A.; Clerk of the Medical Council.
STOTT, WALTER, B.A.; Royal Ins. Co., Liverpl.
STRACHAN, J. C.; Anerley, London, S.E.
STUART, T., M.A., D.Sc.; Plumstead.
SWAMINATHA AIYAR, Professor, M.A.; Madras.
SYMONS, E. W., M.A.; Grove House, Walsall.
TANNER, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dublin.
TATE, JAMES, M.A.; Kedleston Road, Derby.
TAYLOR, Rev. CHARLES, D.D.; Master of St. John's College, Cambridge.
TAYLOR, F. G., M.A., B.Sc.; Nottingham.

TAYLOR, H. M., M.A., F.R.S.; Fellow Trin. Coll., Cambridge.
TAYLOR, J. H., M.A.; Little Trinity, Cambridge.
TAYLOR, W. W., M.A.; Oxford.
TERRY, Rev. T. R., M.A.; Ilsley Rectory.
THIRD, J. A., M.A.; Beith, Ayrshire.
THOMAS, A. E., M.A.; Merton College, Oxford.
THOMAS, Rev. D., M.A.; Garsington Rect., Oxford.
THOMPSON, A. W.
THOMPSON, C. H.; St. David's Coll., Lampeter.
THOMSON, Rev. F. D., M.A.; Ex-Fellow of St. John's Coll., Cambridge.
THOMSON, W. L.; Edinburgh.
THYAGARAGAIYAR, V. R., M.A.; Bangalore.
TIPPLE, E. F.; Thomason Coll., Roorkee.
TIRELLI, Dr. FRANCESCO; Univ. di Roma.
TODD, E. WALTER; Belfast.
TOMBE, A. S., M.A.; Tech. Schools, Plymouth.
TRACHTENBERG, H. L., B.A.; Maida Vale, W.
TRACHTENBERG, M. I.. St. John's Coll., Camb.
TRAILL, ANTHONY, M.A., M.D.; Fell. T.C.D.
TUCKER, R., M.A.
VAN DER HEYDEN, A. F., M.A.; Middlesbrough.
VENKATA, Prof. K. RAMANAIYA; Madras.
VIGARIÈ, ÉMILE; Laissac, Aveyron.
VINCENZO, JACOBINI; Università di Roma.
VOSE, Professor G. B.; Washington.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, G. F.; Queens' College, Cambridge.
WALLIS, B. C., F.C.P., B.Sc. (Econ.); Stamford Hill, N.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARBURTON-WHITE, R., B.A.; Salisbury.
WARD, BEATRICE A., B.Sc.; Cheltenham.
WARD, F. L., B.A.; Hartlepool.
WARREN, A. T., M.A.; Usk Terrace, Brecon.
WARREN, R., M.A.; Trinity College, Dublin.
WASSON, J. C.; Kells, Meath.
WATHERSTON, Rev. A. L., M.A.; Bowdon.
WATSON, D., M.A.; Folkestone.
WATSON, Rev. H. W.; Ex-Fell. Trinity Coll., Camb.
WATTS, J. O.; Middlesbrough.
WEBB, G. H.; Halifax.
WEBB, H. A.; Trinity College, Cambridge.
WEBSTER, H. W., M.D.; Holmesdale Road, Kew Gardens.
WELLACOT, Rev. W. T., M.A.; Newton Abbot.
WEETSCH, FRANZ; Weimar.
WEST, J. WOODCOTE; New Cross.
WHALLEY, L. J., B.Sc.; Leytonstone.
WHAPHAM, R. H. W., M.A.; Manchester.
WHITAKER, R. J.; Stockton-on-Tees.
WHITE, E. A., M.A.; Eckington, Pershore.
WHITEHEAD, R. F., B.A.; London.
WHITESIDE, G. M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. Preb. W. A., M.A.; London.
WIGGINS, Rev. T., B.A.; St. Beuno's Coll., St. Asaph,
WILCOX, Rev. A. M., M.A.; Branhope, Leeds.
WILKINSON, J. F.; Bacup.
WILLIAMS, A. E., M.A.; Texas.
WILLIAMS, D. J., M.A.; Worcester Coll., Oxon.
WILLIAMSON, B., M.A.; F. & T. Trin. Coll., Dub.
WILMOT, Rev. W. T., M.A.; Newton Abbot.
WILSON, Rev. J., M.A.; Bannockburn.
WILSON, Ven. Archdeacon J. M., M.A., F.G.S.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOLSTENHOLME, Professor, M.A., Sc.D.
WOLTERS, J. B.; Groningen.
WOODALL, H. J., A.R.C.S.; Stockport.
WOODCOCK, T., B.A.; Twickenham.
WORKMAN, W. P., M.A.; Bath.
WORTHINGTON, W. R., M.A.; Southport.
WRIGHT, W. J.; New Jersey, U.S.A.
YOUNG, JOHN, B.A.; Portadown, Ireland.
YOUNGMAN, C. E., M.A.; Corinth House, Cheltenham.
ZERR, Professor, M.A.; Staunton, Virginia.

*** Of this Mathematical publication there have now been issued the seventy-five Volumes of the Original Series and thirteen Volumes of the New Series. Each Volume contains, in addition to the papers and solutions that have appeared in *The Educational Times*, an equal quantity of new matter, consisting of further papers and solutions, and of completions of, or additions to, papers and solutions that have been partly published in the pages of the journal.

These volumes contain contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

CONTENTS.

Mathematical Papers, &c.

Page.

Compound Interest Problems; Note on the Empirical Rule of Mr. Saradakanta Ganguli, M.A. (Professor Sanjána, M.A.) 40

A Demonstration of the Transcendental Character of the Function $e^{H(x)}$. (F. Tavani) 50

Given Four Tangents to a Conic, to find the Locus of its Centre. (A new solution by Saradakanta Ganguli, M.A.) 91

Note on Expansion in Series. (S. Narayana Aiyar) 49

Note on Factorization of $N = x^8 + y^8$. (Lt.-Col. Allan Cunningham, R.E.) 97

Note on Freedom and Geometry. (S. C. Bagchi, B.A., LL.B.) ... 33

Note on the Nine-Point Circle. (W. Gallatly, M.A.) 105

Remarks on Dr. Stuart's Method of Factorization of 8-vans. (Lt.-Col. Allan Cunningham, R.E.) 98

Solved Questions.

7507 (R. Knowles, B.A., L.C.P.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{2\cos^2 a - \sin^2 2a \cos^2 x}{1 - \sin^2 2a \cos^2 x} \log(\cos x)\, dx = \pi \log\left(\tfrac{1}{2}\sec a\right) \quad \ldots \quad 70$$

9000. (R. Knowles, B.A.)—In any triangle prove that

$$a\cos(B-C) + b\cos(A-C) + c\cos(A-B) = 4/R. \quad \ldots\ldots \quad 68$$

9615. (Professor Bordage.)—Find (1) the most general functions f and ϕ such that for all values of x and y the expression

$$V = (x^2 - y^2)\,[f(x+y) + \phi(x-y)]$$

can be put under the form $V = F(x) - \Phi(y)$; and (2) the functions F and Φ 38

9992. (Professor Emmerich, Ph.D.)—Construct a triangle, having given the vertices A_1, B_1, C_1 of its first Brocard triangle 88

Page.

10502. (H. W. Segar.)—(1) show that the curve
$$r^4 - 2c^2r^2\cos 2\theta + c^4 = 4a^2c^2\sin^2\theta$$
has two conjugate points other than the origin, and the normals drawn from them are of length $2a$; and (2) draw a figure of the curve ... 107

10599. (Professor Gram.)—Résoudre l'équation
$$[abx(x-a-b)]^{\frac{1}{2}} + [bcx(x-b-c)]^{\frac{1}{2}} + [cax(x-a-c)]^{\frac{1}{2}} = [abc(a+b+c)]^{\frac{1}{2}}$$
..... 104

10775. (J. L. Mackenzie, B.A., B.Sc.) — Two circles cut a given circular cubic in A, B, C, D and A, B, E, F respectively; CE cuts the cubic in P, PD in Q, QF in R. Prove that DR touches the cubic at D. 90

10841. (The late Professor Clifford, F.R.S.) — If, in regard to a system of n quadric surfaces, the two systems of n polar planes in regard to any two points of space are projective to one another, either the quadrics have a common Jacobian or each of them is a double plane 89

11110. (R. W. D. Christie.)—Find, by means of the quintic roots of minus unity, all the values in succession of X and Y in the equation $X^5 - 5Y^5 = \pm 4$, and give the values of X and Y respectively 35

11215. (Professor Lemoine.)—Sur les perpendiculaires MA', MB', MC' abaissées d'un point M sur les côtés du triangle ABC on prend respectivement les points A'', B'', C'' tels que
$$MA'.MA'' = MB'.MB'' = MC'.MC'' = \rho^2.$$
Quel que soit ρ, le triangle A''B''C'' est perspectif avec ABC et le centre de perspective est sur l'hyperbole équilatère ABCM 67, 93

11643. (R. W. D. Christie.)—Show how to arrange the signs so as to make the following equations true :—
$$1^n \pm 2^n \pm 3^n \pm \ldots \pm (2^{n+1})^n = 1^n \pm 3^n \pm 5^n \pm \ldots \pm (2^{n+2}-1)^n = 0.$$
Ex. gr. : $-1^3 + 2^3 + 3^3 - 4^3 + 5^3 - 6^3 - 7^3 + 8^3$
$$= -1^3 + 3^3 + 5^3 - 7^3 + 9^3 - 11^3 - 13^3 + 15^3 = 0$$
..... 110

12236. (J. J. Walker, F.R.S.)—If x, y, z satisfy
$$lx + my + nz = 0 \quad \text{and} \quad \Sigma mnx\left\{3u\frac{d^2u}{dx^2} - 2\left(\frac{du}{dx}\right)^2\right\} = 0,$$
being any ternary cubic form, show that
$$2lmn\,\Sigma lyz\left(3u\frac{d^2u}{dy\,dz} - 2\frac{du}{dy}\frac{du}{dz}\right) = \Sigma m^2n^2x(y+z)\left\{3u\frac{d^2u}{dx^2} - 2\left(\frac{du}{dx}\right)^2\right\}$$
..... 19

15254. (Professor Sanjána, M.A.)—On the sides AB, AC ($c > b$) of the triangle of reference similar segments of circles are described towards BC. Prove that the locus of the other point of intersection of these circles is the nodal circum-cubic
$$\alpha(\beta^3 - \gamma^3) = 2\beta\gamma(\gamma\cos C - \beta\cos B).$$
Show that the tangents to the cubic at B and C and the line joining A to the circum-centre of the triangle all meet the cubic at the same point, and that the satellite of BC touches the curve, and is given by
$$\alpha(\cos^2 B - \cos^2 C) = 2(\beta\cos^3 C - \gamma\cos^3 B).$$
Also find the point of inflexion on the cubic 68

Page.

15541. (Rev. T. Roach, M.A.)—Sum the series

(1) $\frac{1}{1.2}-\frac{1}{5.6}+\frac{1}{9.10}-\ldots$, (2) $\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\ldots$ ... 46

15874. (N. Parameswara Menon.) — ABC is a given triangle; through A, A_1B_1 is drawn parallel to the perpendicular from B on AC; through B, B_1C_1 is drawn parallel to the perpendicular from C on AB; and, through C, C_1A_1 is drawn parallel to the perpendicular from A on BC. Find (i.) the area of the triangle $A_1B_1C_1$ in terms of that of the triangle ABC; (ii.) the circum-radius of the triangle $A_1B_1C_1$; (iii.) the area of the pedal triangle of $A_1B_1C_1$ in terms of that of the pedal triangle of ABC; and (iv.) show that, if A_1B_1 cut the circum-circle of ABC in A_2, and B_2, C_2 are the two other points obtained in the same way, the triangle $A_2B_2C_2$ is identically equal to the triangle ABC ... 22

15994. (Hon. G. R. Dick, M.A.)—Quadrics are drawn through seven given points, four of which are taken as the vertices of the tetrahedron of reference. If (x_1, y_1, z_1, w_1), ..., (x_3, y_3, z_3, w_3) are the co-ordinates (quadriplanar or tetrahedral) of the other three given points, and the notation $|\,a_1\,b_2\,c_3\,|$ is employed to denote the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then the co-ordinates of the eighth point through which all the quadrics of the system pass are given by

$$x\,\frac{|\,z_1w_1,\ y_2w_2,\ y_3z_3\,|}{|\,y_1,\ z_2,\ w_3\,|} = \text{similar quantities} \quad \ldots\ldots\ 66$$

16007. (Professor Nanson.) —Eliminate x, y from

$$x^2 = ax+by+c, \quad xy = a'x+b'y+c', \quad y^2 = a''x+b''y+c'' \quad \ldots\ 47$$

16014. (R. F. Davis, M.A.)—Where is there to be found an investigation of the locus of the focus of a parabola which passes through three given points? 101

16041. (Professor Sanjána, M.A., and M. V. Arunachalan, M.A.)—If $R_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+1/n$, prove that

$$R_1-R_2+R_3-R_4+\ldots \text{ad inf.} = \tfrac{1}{2}\log 2 \quad \ldots\ldots\ 17, 42$$

16071. (Professor Neuberg.)—D'un point donné P on peut mener à une parabole Σ trois normales PM, PM', PM''; les points d'incidence M, M', M'' sont situés sur une hyperbole équilatère Σ' passant par P. Etant donnés cette hyperbole et les points M, M', construire le point M'', le point P et le sommet de la parabole 44

16077. (R. W. D. Christie.)—It is known that

$$(10^{32}+1) = 26 \pmod{97}.$$

It is required by the substitution of units for ciphers to find a square residue 49 to the same modulus 76

16088. (Professor Nanson.) — Find the condition that two given conics may intersect a given line in the same two points 87

16090. (H. Bateman, B.A.)—A_1, A_2, A_3, A_4 are any four points in a plane. Out of these any point such as A_4 is chosen, and lines are drawn through A_1, A_2, A_3 perpendicular to A_4A_1, A_4A_2, and A_4A_3 respectively, so as to form a new triangle $B_1B_2B_3$. Prove that the ratio of the areas of the triangles $A_1A_2A_3$, $B_1B_2B_3$ is the same whichever point A_4 is chosen 72, 91

Page.

16097. (W. J. Greenstreet, M.A.)—Show that

$$\frac{b^2}{ab+a-}\ \frac{ab}{a+b+ab-}\ \frac{ab}{a+b+ab-}\ \cdots$$

to n quotients, identical after the first, is equal to $\frac{b}{a+}\ \frac{a}{b+}\ \frac{b}{a+}\ \frac{a}{b+}\ \cdots$ to $2n$ quotients. 56

16101. (Professor Neuberg.)—Soient A′, B′, C′, D′ les projections des sommets A, B, C, D sur un plan quelconque. Démontrer que les perpendiculaires abaissées des points A′, B′, C′, D′ respectivement sur les faces BCD, CDA, DAB, ABC du tétraèdre sont quatre génératrices d'un même hyperboloïde 51

16110. (James Blaikie, M.A.) — Trace from geometrical considerations the locus of a point P which moves so that two given straight lines AB, CD subtend the same angle at the point, considering the cases where ABCD is (1) a square, (2) a rectangle, (3) a parallelogram, (4) an irregular quadrilateral 102

16118. (T. Stuart, D.Sc.)—Solve in positive integers the equation $x^3+y^3=53z^3$. Is the equation $x^3+y^3=66z^3$ possible in integers? 27

16123. (W. F. Beard, M.A.) — If a circle be drawn through the focus of a parabola to touch it at P and cut it at Q, R, then QR passes through a fixed point 80

16131. (D. Biddle.)—At the vertices of an equilateral triangle are weights of value p, q, r $(p+q=r)$, which travel along the sides in the same positive direction at a uniform pace. Trace the locus of the centre of gravity. Also extend the investigation to triangles which are not equilateral, on the understanding that, though the sides be unequal, they are traversed by the weights in equal times 92

16143. (Professor Neuberg.)—On considère les hyperboles équilatères qui passent par deux points fixes A, B et dont une asymptote passe par une point fixe C. Trouver (1) le lieu des centres de ces courbes; (2) l'enveloppe des secondes asymptotes 21

16152. (M. V. Arunachalan, M.A.)—Show that

$$[a(b-c)(bc-aa')(a^4-a'^4)]/(a-a') + [b(c-a)(ca-bb')(b^4-b'^4)]/(b-b') + [c(a-b)(ab-cc')(c^4-c'^4)]/(c-c')$$
$$= (b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,S/abc,$$

where $aa'=bb'=cc'$ and $S=a+b+c+a'+b'+c'$. 32

16156. (R. F. Davis, M.A.)—Prove that the equation to a parabola touching the oblique axes Ox, Oy, whose focus is at the point (ξ, η), is $\sqrt{(x\xi)}+\sqrt{(y\eta)}=\sqrt{(\xi^2+2\xi\eta\cos\omega+\eta^2)}$ where $\omega=\angle xOy$ 18

16158. (V. Ramaswami Aiyar, M.A.)—ABC is a triangle inscribed in a conic (focus S). If the base BC passes through the reflection of A with respect to the S directrix, then the pedal circle of the triangle ABC with respect to S has its diameter equal to SA 75

16176. (Professor Lauvernay.)—Construire la courbe

$$\rho^2-4\rho\cos\tfrac{1}{2}\omega+1=0$$ 55

Page.

16183. (Professor R. W. Genese, M.A.)—Inward forces act at the mid-points of the sides of any plane quadrilateral at right angles and proportional to the opposite sides. Prove (1) that they balance, (2) that the resultant of two opposite forces acts along the directrix of the parabola inscribed in the quadrilateral 58

16185. (Professor Neuberg.)—Trouver

$$\int \frac{\sin 5x}{\sin 3x}\,dx, \quad \int \frac{\sin 3x}{\sin 5x}\,dx. \quad \text{..... } 36$$

16187. (V. Ramaswami Aiyar, M.A.)—A. and B. play at a game, in which each game is won by A. or B. by an equal chance, for a fixed stake on each game. B. has enough money; while A., having none, poses as if he has enough, too. Show that the probability that A. would be able to maintain his pretension for $2n-1$ games or $2n$ games is $\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\ldots\,\frac{2n-1}{2n}$ 65

16188. (Lt.-Col. Allan Cunningham, R.E.)—Determine the sides of two right-angled triangles in integers so that the product of the hypothenuse and a side of one may be equal to the product of the hypothenuse and a side of the other 24

16189. (T. Stuart, M.A., D.Sc.)—Find a direct, non-tentative, and quick process for solving the equation $x^2 = a \pmod p$ where a is a quadratic residue of p and p is a large prime. Exhibit your method by solving the equations

(*a*) $x^2 = -537 \pmod{6311}$, (*b*) $x^2 = 3582 \pmod{7177}$,
(*c*) $x^2 = 3789 \pmod{8513}$ 19

16190. (R. W. D. Christie.)—If N = AB, all integral, prove that the small factor $A = \{[4bN + (a-bc)^2]^{\frac{1}{2}} - a + bc\}/2b$ and the large factor $B = \frac{1}{2}\{[4bN + (a-bc)^2]^{\frac{1}{2}} + a - bc\}$. Also $(a-bc)^2 = (B-bA)^2$, all the letters being integral: *e.g.*, N = 257821. Then one set of values (out of many) is $b = 2$, $c = 10$, $a = 69$; or, again, $b = 2$, $c = 8$, $a = 65$. It is required to find the small letters 42, 62

16192. (G. Bool, B.A.)—Find the solution of

$$a^2 - b^2 = b^2 - c^2 = 13 \quad \text{..... } 52$$

16197. (L. Isserlis, B.A.)—A straight line moves so as to cut three fixed conics in six points forming an involution. Find the locus of the centre of the involution 26, 93

16199 (W. F. Beard, M.A.)—Find the envelope of a straight line on which two given circles intercept equal chords 84

16200. (Professor Sanjána, M.A.)—The great circle bisecting perpendicularly the side BC of the spherical triangle ABC meets the great circles bisecting internally and externally the angle A in the points D, E. Prove that D and E lie on the small circle circumscribing ABC 58

16203. (R. F. Scott, M.A.)—Find a solution of the system of n equations $x_1 - a_1 = x_2 - a_2 = \ldots = x_n - a_n$,

$$\begin{vmatrix} x_1, & x_2, & \ldots, & x_n \\ x_n, & x_1, & \ldots, & x_{n-1} \\ \ldots & \ldots & & \ldots \\ x_2, & x_3, & \ldots, & x_1 \end{vmatrix} = 0 \quad \text{..... } 54$$

Page.

16205. (Lt.-Col. Allan Cunningham, R.E.) — Given that $N = x^x$ has 38907 figures, x being an integer, find its first three and last three figures 34

16206. (T. Stuart, M.A., D.Sc.)—The number 10091401 has been proved prime by both Euler and Legendre, but only after tedious calculations. Obtain the same result by calculations not requiring more than an hour's work 43

16208. (Professor Nanson.)—If

$a^n + b^n + c^n = 0$, $a^{n-1}x + b^{n-1}y + c^{n-1}z = 0$, $x^n + y^n + z^n = 0$, and $abc \neq 0$,

then either x/a, y/b, z/c are all equal or the sum of their homogeneous products of $n-2$ dimensions is zero 40

16209. (J. J. Barniville, B.A., I.C.S.)—Prove that

$$\frac{1}{1.3} + \frac{1.2}{3.7} + \frac{3.2^2}{7.17} + \frac{7.2^3}{17.41} + \ldots = 1;$$

$$\frac{1}{1.3} + \frac{2.2}{3.8} + \frac{5.2^2}{8.21} + \frac{13.2^3}{21.55} + \ldots = 1 \text{ ..... } 59$$

16210. (A. M. Nesbitt, M.A.) — Prove from first principles that the locus of a point whence a straight line can be drawn to intersect $(x-a_r)/l_r = (y-\beta_r)/m_r = (z-\gamma_r)/n_r$ $(r = 1, 2, 3)$ is $|\, P_1, Q_2, R_3 \,| = 0$, where P_k, Q_k, R_k are the co-factors of p_k, q_k, r_k in the determinant

$$\begin{vmatrix} p_k, & q_k, & r_k \\ l_k, & m_k, & n_k \\ x-a_k, & y-\beta_k, & z-\gamma_k \end{vmatrix}.$$

[Note that the locus is a quadric and that the three given lines are obviously generators.] 48, 59

16212. (W. J. Greenstreet, M.A.) — Find the locus of the feet of the perpendiculars from the origin to the normals to Bernoulli's lemniscate 18, 51

16213. (R. N. Apte.)—The normals at P, Q of a parabola are at right angles, and from their point of intersection the third normal is drawn and meets the curve in R. Find the least distance of the centroid of the triangle PQR from the vertex. Also find the least distance from the vertex of the centroid of the triangle formed by tangents at P, Q, R 53

16216. (W. F. Beard, M.A.)—Two similar ellipses with their axes parallel are cut by any straight line in A, B, A′, B′; a point O is taken in AD such that $OA.OB = OA'.OB'$. Prove that the locus of O is a straight line 21, 33

16218. (Professor Lauvernay.)—Construire un triangle, connaissant les rapports a/b, b/c et la somme $m_a + m_b + m_c$ des médianes 54

16219. (S. Narayana Aiyar.)—In the reduction of the differential expression $dx/\sqrt{(MN)}$ to the standard form, the values given by Cayley in case (4), $MN = (1+mx^2)(1+nx^2)$ where $m > n$ for x^2, k^2, and $dx/\sqrt{(MN)}$ are $1/m \,.\, [y^2/(1+y^2)]$, $(m-n)/n$ and $(1/\sqrt{m}).(dy/\sqrt{Y})$ (Art. 414, Cayley's *Elliptic Functions*, Second Edition). Show that these values are incorrect and find the correct values 68

Page.

16220. (Hon. George R. Dick, M.A.)—Show that the determinant

$$\begin{vmatrix} \omega_1+\omega_2-\omega_4-\omega_5, & \omega_1\omega_2-\omega_4\omega_5, & \omega_1\omega_2(\omega_4+\omega_5)-\omega_4\omega_5(\omega_1+\omega_2) \\ \omega_2+\omega_3-\omega_5-\omega_6, & \omega_2\omega_3-\omega_5\omega_6, & \omega_2\omega_3(\omega_5+\omega_6)-\omega_5\omega_6(\omega_2+\omega_3) \\ \omega_3+\omega_4-\omega_6-\omega_1, & \omega_3\omega_4-\omega_6\omega_1, & \omega_3\omega_4(\omega_6+\omega_1)-\omega_6\omega_1(\omega_3+\omega_4) \end{vmatrix}$$

vanishes identically. By attributing an appropriate geometrical significance to the ω's, deduce the truth of Brianchon's theorem .. 45

16221. (Lt.-Col. Allan Cunningham, R.E.)—Prove that

$$(x+1)^{x+1}-(x^2+x\cdot 1) \equiv 0 \bmod x^3 \text{ if } x \text{ be odd}$$

and $\equiv 0 \bmod \frac{1}{2}x^3$ if x be even 86

16223. (T. Stuart, M.A., D.Sc.)—Show that there is a doubly infinite series of numbers of the form X^8+Y^8 which are algebraically resoluble into two factors; and find the general forms of X and Y when this is possible. Factorize 5^8+17^8, 19^8+7^8, 19^8+23^8, 29^8+25^8 95, 97

16224. (Professor Sanjána, M.A.)—Prove that

$$1\,\frac{1}{2}+\frac{1}{2}\,\frac{1.3}{2.4}+\frac{1}{3}\,\frac{1.3.5}{2.4.6}+\frac{1}{4}\,\frac{1.3.5.7}{2.4.6.8}+\ldots=\log_e 4,$$

$$\frac{1}{2}\,\frac{1}{2}+\frac{1}{3}\,\frac{1.3}{2.4}+\frac{1}{4}\,\frac{1.3.5}{2.4.6}+\ldots=1,$$

$$\frac{1}{3}\,\frac{1}{2}+\frac{1}{4}\,\frac{1.3}{2.4}+\frac{1}{5}\,\frac{1.3.5}{2.4.6}+\ldots=\frac{5}{6}$$

and show how to find the sums of similar series 38

16229. (Professor Nanson.)—Prove geometrically that two concentric conics have a pair of conjugate diameters in common, unless their asymptotes are real and interlaced 53

16230. (Professor Cochez.)—On considère toutes les coniques passant par quatre points donnés. Démontrer que les polaires d'un point fixe par rapport à toutes ces coniques passent au même point 41, 77

16232. (D. Biddle.)—A square holding has been staked out on a level part of the prairie. But a fire has invaded the scene, and all traces of the square have vanished, except the charred remains of four of the stakes, known to have been on distinct sides of it. At present they form the vertices of an irregular quadrilateral. One of the owners, being a mathematician, says that the data are sufficient to enable him to stake out the square afresh on its original lines. How so? 86

16233. (W. F. Beard, M.A.)—The radical axis of two circles is midway between the polars of the centres of each circle with respect to the other 94

16241. (Saradakanta Ganguli, M.A.)—If the Pythagorean definition of numbers in musical progression be extended to any numbers, positive or negative, integral or fractional, rational or irrational, find a series of numbers such that, if any four consecutive numbers of the series be taken, they will be in musical progression. Show that there are only two such principal series, and that all other series in musical progression are deducible from them 75

16242. (Lt.-Col. Allan Cunningham, R.E.)—Factorize completely (into prime factors) $N = (3^{54}+2^{54})$ 63, 106

16244. (R. F. Davis, M.A.)—If x, y are positive quantities, prove that each of them is less than $(x^2+xy+y^2)^2/[4xy(x+y)]$ 81

Page.

16248. (W. J. Dobbs, M.A.)—The edges of a tetrahedron are of lengths a, a, b, b, c, c, opposite edges being equal. Show that its volume is $\frac{1}{12}\sqrt{[2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)]}$ 99

16249. (W. J. Greenstreet, M.A.)—OQ is the perpendicular from the origin O to the tangent to a curve at P. If C is the centre of curvature at P, and D the foot of the perpendicular from C to OP, E, M the feet of the perpendiculars from D, O to CP, then shall QM and OE intersect in the centre of curvature at Q 70

16251. (W. F. Beard, M.A.)—If a system of conics have a common chord, then the polars of any point on the common chord, with regard to the conics, are concurrent 59

16252. (James Blaikie, M.A.)—If a, b, c are the sides of a triangle ABC; p, q, r the distances of a point D from A, B, C, prove that A, B, C, D are coplanar when the following relation holds good:—

$$\Sigma[(a^2p^2+b^2c^2)(q^2+r^2)] = \Sigma[a^2p^2(a^2+p^2)] + \Sigma a^2q^2r^2 + a^2b^2c^2.$$

Show, both geometrically and algebraically, that this condition is identical with

$$\Sigma[(a^2p^2+q^2r^2)(b^2+c^2)] = \Sigma[a^2p^2(a^2+p^2)] + \Sigma a^2q^2r^2 + a^2b^2c^2 \text{ ... } 69$$

16253. (C. M. Vijendra Rao.)—To draw a straight line cutting any two sides of a given triangle such that the intercept of the former between the latter may be equal to each of the intercepts of the latter between the base and the required straight line 81

16255. (Alfred A. Robb.)—The solution of the differential equation $d^2y/dx^2 = Ay^m x^n$ may be reduced to quadratures when either

$$n = -(m+3) \quad \text{or} \quad n = -\tfrac{1}{2}(m+3).$$

Show that the equation may also be solved when

$$m = 2,\ n = -\tfrac{15}{7};\quad m = 2,\ n = -\tfrac{20}{7} \text{ .....(1; 2)}$$
..... 81

16257. (G. H. Hardy, M.A.)—Show that the series

$$\log 1 - \log 2 + \log 3 - \ldots$$

is summable, *i.e.*, that, if s_n denotes the sum of the first n terms, then $(s_1+s_2+\ldots+s_n)/n$ tends to a limit as n tends to infinity: and find its sum. Show also that, if $\phi(n)$ tends steadily to 0 as n tends to infinity, the series $R_1 - R_2 + R_3 - \ldots$ where $R_n = \phi(1)+\phi(2)+\ldots+\phi(n)$ is summable, and its sum is equal to that of the convergent series

$$\phi(1) - \phi(2) + \phi(3) - \ldots \text{ .... } 79$$

16260. (R. F. Davis, M.A.) — If $x-M$, $x-N$, $x-(2M-N-2)$, $x-(2N-M-2)$ are all perfect squares, find x in terms of M and N
..... 106

16261. (Robt. W. D. Christie.)—Solve

$$x^2/z^2 - y^2/z^2 = y^2/z^2 - w^2/z^2 = 65,$$

and give an example where there are two sets of values such as

$$\left(\frac{85}{6}\right)^2 - \left(\frac{41}{6}\right)^2 = \left(\frac{113}{6}\right)^2 - \left(\frac{85}{6}\right)^2 = \left(\frac{373}{30}\right)^2 - \left(\frac{23}{30}\right)^2$$
$$= \left(\frac{527}{30}\right)^2 - \left(\frac{373}{30}\right)^2.$$

Can we substitute n = any integer for 65? 77

Page.

16262. (W. P. Workman.) — If the fraction $\frac{1}{7}$ be converted into a radix fraction in scales 2, 3, 4, 5, 6, ..., the numbers of figures in the period are respectively 3, 6, 3, 6, 2, 1, 1, 3, 6, 3, 6, 2, 1, 1, ..., the numbers forming a repeating group of 7 terms. Prove this and show that the same law holds for a fraction 1/N, whether N be prime or not, if, in the latter case, the number of figures completely expressing the radix fraction be taken 71

16263. (Professor Sanjána, M.A.)—Given that the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ are $\alpha^2, \beta^2, \gamma^2, \ldots, \nu^2$, is there any simple process by which equations of the same degree with roots $\alpha, \beta, \gamma, \ldots, \nu$ and $-\alpha, -\beta, -\gamma, \ldots, -\nu$ can be written down? [*E.g.*, the roots of $64x^3 - 112x^2 + 56x - 7 = 0$ are $\sin^2\frac{2}{7}\pi$, $\sin^2\frac{4}{7}\pi$, $\sin^2\frac{8}{7}\pi$; the equations with the square roots taken positively and negatively as roots are $8x^3 - 4\sqrt{7}x^2 + \sqrt{7} = 0$, $8x^3 + 4\sqrt{7}x^2 - \sqrt{7} = 0$] 109

16264. (Professor Neuberg.)—Le plan tangent en un point quelconque de la surface $xyz = k$ rencontre les axes coordonnés rectangulaires Ox, Oy, Oz aux points A, B, C. Trouver le lieu de l'orthocentre du triangle ABC 103

16265. (James Blaikie, M.A.) — ABCD is a tetrahedron in which AB.CD = AC.BD = AD.BC. Prove that the 12 bisectors of the interior angles of the triangular faces intersect in 6 points on the edges of the tetrahedron, and that the straight lines joining the points of intersection lying on opposite edges are concurrent; also that the 12 bisectors of the exterior angles meet in 6 points on the edges produced, and that these 6 points are in one plane 88

16267. (A. M. Nesbitt, M.A.)—A variable conic, which is (*a*) an equilateral hyperbola, (*b*) a parabola, is drawn to have four-pointic contact with the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Show that the locus of the centre of (*a*) is a bicircular quartic which is the inverse of the fixed hyperbola with respect to a concentric circle of radius $\sqrt{(a^2 - b^2)}$; and that the locus of the focus of (*b*) is also a bicircular quartic, which is its inverse with respect to a concentric circle of radius $\sqrt{(a^2 + b^2)}$ 111

16270. (L. Isserlis, B.A.) — An ellipse of semi-axes a, b passes through two fixed points A, B, a distance $2d$ apart. Prove that the locus of its centre is a sextic curve, which has four nodes on the circle on AB as diameter, if $\frac{1}{2}\sqrt{(a^2 + b^2)} > d > ab/\sqrt{(a^2 + b^2)}$. The nodes degenerate into cusps if $d = \frac{1}{2}\sqrt{(a^2 + b^2)}$, and the tangents at the cusps are parallel to AB 95

16272. (C. Joss, M.A.)—Prove that the nine-point circle passes through twelve uniformly related points 88

16276. (C. M. Ross.)—If

$$\begin{vmatrix} \sin 2\theta, & \cos\theta, & \sin\theta, & 1 \\ \sin 2\phi, & \cos\phi, & \sin\phi, & 1 \\ \sin 2\psi, & \cos\psi, & \sin\psi, & 1 \\ \sin 2\chi, & \cos\chi, & \sin\chi, & 1 \end{vmatrix} = 0$$

and if θ, ϕ, ψ, χ are different positive angles each less than 2π, prove that $\theta + \phi + \psi + \chi = n\pi$ (n an odd integer) 107

16279. (Robt. W. D. Christie.)—Solve by an easy general process, integrally or fractionally, $X^3 - pY^3 = 1$ where in the former case Y is > 1, as, *e.g.*, $18^3 - 17.7^3 = 1$, $\left(\frac{36}{17}\right)^3 - 19\left(\frac{13}{17}\right)^3 = 1$ 90

Page.

16280. (Professor A. Droz-Farny.)—Si P_λ/Q_λ est une réduite de rang λ de la fraction continue $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}$, démontrer que

$$P_\lambda = Q_{\lambda-1} + Q_\lambda, \qquad P_\lambda = P_{\lambda-1} + 2Q_{\lambda-1} \quad \ldots\ldots\ldots\ 108$$

16292. (*Communicated by* H. MacColl, B.A.)—"*Whatever P and Q may stand for, we may show, à priori, that some P is Q.* For *All PQ is Q* by the law of identity, and similarly *All PQ is P*; therefore, by a syllogism in *Darapti*, *Some P is Q.*"—How would you deal with this paradox? [From Dr. Keynes's *Formal Logic.*] 98

16301. (W. J. Greenstreet, M.A.)—If r', ρ' are radius vector and radius of curvature of the inverse of a curve corresponding to r, ρ, find $r/\rho + r'/\rho'$ in terms of the angle between the tangent and radius vector of the curve or its inverse..... 104

ERRATA.

Page 21, line 10, *for* "of centre" *read* "of the centre."

Page 52, Solution I. to 16192, line 2, *for* "$b = 6\sqrt{2}$" *read* "$c = 6\sqrt{2}$."

Page 100, Solution III., fourth line from end, *for* "^{1} base" *read* "$\frac{1}{3}$ base."

Page 101, last line but one, *for* "th" *read* "that."

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

16041. (Professor SANJÁNA, M.A., and M. V. ARUNACHALAN, M.A.)—If $R_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n$, prove that

$$R_1 - R_2 + R_3 - R_4 + \ldots \text{ ad inf.} = \tfrac{1}{2}\log 2.$$

Notes (I.) *by* R. J. WHITAKER; (II.) *by* L. ISSERLIS, B.A.

(I.) If $S \equiv R_1 - R_2 + R_3 - R_4 + \ldots$ ad inf., then, taking an odd number of terms,

$$S = R_1 + (R_3 - R_2) + (R_5 - R_4) + \ldots = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots \text{ ad inf.} \quad \ldots \text{ (i.).}$$

Also, taking an even number,

$$S = -\tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} - \ldots \text{ ad inf.} \quad \ldots\ldots\ldots\ldots \text{ (ii.),}$$

wherefore $2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \log 2$ and $S = \frac{1}{2}\log 2$.

Note.—(i.) and (ii.) are both divergent; so it would appear to be more accurate to say that S is an alternating series whose sum is $+\infty$ or $-\infty$, according as the number of terms is odd or even, and that its values alternate at distances from $\frac{1}{2}\log 2$ which, though they are infinite, tend to become equal to one another.

(II.) Apparently the series $R_1 - R_2 + \ldots$ is not convergent. For the limit when $n = \infty$, to which R_n tends, is ∞. Hence, if ${}_mS_n$ = sum of m terms immediately following the nth, $\underset{n=\infty}{\mathrm{L}}\, {}_mS_n = \infty$ or 0, according as m is odd or even, and therefore the series is divergent, or rather oscillates; so that the sum of an even number of terms is finite and that of an odd number is infinite. It would not be legitimate to say

$$\begin{aligned} S = R_1 - R_2 + R_3 - R_4 + \ldots \\ &= \tfrac{1}{2}[2R_1 - 2R_2 + 2R_3 - 2R_4 + \ldots] \quad \ldots\ldots\ldots\ldots \text{ (A)} \\ &= \tfrac{1}{2}[R_1 + (R_1 - R_2) - (R_2 - R_3) + (R_3 - R_4) - \ldots] \quad \ldots \text{ (B)} \\ &= \tfrac{1}{2}[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots] = \tfrac{1}{2}\log 2, \end{aligned}$$

because the series (B) is not necessarily the same as the series (A). (*Chrystal, Algebra*, Part II., chap. xxvi., §§ 8 and 7.)

16156. (R. F. Davis, M.A.)—Prove that the equation to a parabola touching the oblique axes Ox, Oy, whose focus is at the point (ξ, η), is $\sqrt{(x\xi)} + \sqrt{(y\eta)} = \sqrt{(\xi^2 + 2\xi\eta\cos\omega + \eta^2)}$ where $\omega = \angle xOy$.

Solutions (I.) *by* W. J. Dobbs, M.A., *and others*; (II.) *by* M. T. Naraniengar, M.A., *and* F. W. Reeves, M.A.

(I.) The general equation of a third tangent to the parabola $\sqrt{(x/a)} + \sqrt{(y/b)} = 1$ is $x/\lambda + y/\mu = 1$, subject to the condition

$$\lambda/a + \mu/b = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

The equation of the circle circumscribing the triangle formed by the three tangents is $x^2 + 2xy\cos\omega + y^2 = \lambda x + \mu y$. As such a circle always passes through the focus (ξ, η), we must have

$$\lambda\xi + \mu\eta = \xi^2 + 2\xi\eta\cos\omega + \eta^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

for all values of λ and μ subject to (1).

Comparing the equations (1) and (2), we see that

$$a\xi = \xi^2 + 2\xi\eta\cos\omega + \eta^2 = b\eta;$$

and, substituting for a and b, the equation of the parabola is obtained in terms of the constants ξ and η.

(II.) Draw SN ordinate; then

$$\angle OAS = \angle SOB = \angle OSN;$$

therefore OSN and OAS are similar, and

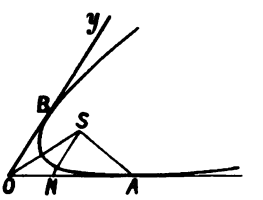
$$OA.ON = OS^2.$$

Similarly, $OB.SN = OS^2$.

Hence the equation to the parabola is

$$\sqrt{(x/OA)} + \sqrt{(y/OB)} = 1;$$

i.e., $$\sqrt{(x\xi)} + \sqrt{(y\eta)} = OS = \sqrt{(\xi^2 + \eta^2 + 2\xi\eta\cos\omega)}.$$



16212. (W. J. Greenstreet, M.A.)—Find the locus of the feet of the perpendiculars from the origin to the normals to Bernoulli's lemniscate.

Solution by S. Narayanan.

The equation of the normal to $r^2 = a^2\cos 2\theta$ is seen to be

$$r\sin(3\alpha - \theta) = a\sin 2\alpha\cos^{\frac{1}{2}} 2\alpha,$$

or $$x\sin 3\alpha - y\cos 3\alpha = a\sin 2\alpha\cos^{\frac{1}{2}} 2\alpha \quad \ldots\ldots\ldots\ldots \quad (i.).$$

The equation of the line at right angles to the above and through the origin is $$y\sin 3\alpha + x\cos 3\alpha = 0 \quad \ldots\ldots\ldots\ldots \quad (ii.).$$

Hence for the intersection of (i.) and (ii.) we have

$$x^2 + y^2 = a^2\sin^2 2\alpha\cos 2\alpha.$$

Hence for all normals the locus of the foot of the perpendicular from the origin is given by

$$x^2 + y^2 = a^2\sin^2 2\theta\cos 2\theta, \quad i.e., \quad r^2 = a^2\sin^2 2\theta\cos 2\theta;$$

or, in Cartesians, $$(x^2 + y^2)^4 = 4a^2x^2y^2(x^2 - y^2).$$

12286 (J. J. WALKER, F.R.S.)—If x, y, z satisfy

$$lx+my+nz=0 \quad \text{and} \quad \Sigma mnx\left\{3u\frac{d^2u}{dx^2}-2\left(\frac{du}{dx}\right)^2\right\}=0,$$

being any ternary cubic form, show that

$$2lmn\,\Sigma lyz\left(3u\frac{d^2u}{dy\,dz}-2\frac{du}{dy}\frac{du}{dz}\right)=\Sigma m^2n^2x(y+z)\left\{3u\frac{d^2u}{dx^2}-2\left(\frac{du}{dx}\right)^2\right\}.$$

Solution by S. SIRCOM.

From Euler's equations,

$$x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}=3u, \quad x\frac{d^2u}{dx^2}+y\frac{d^2u}{dx\,dy}+z\frac{d^2u}{dx\,dz}=2\frac{du}{dx},$$

and two others, we obtain

$$2yz\frac{d^2u}{dy\,dz}=x^2\frac{d^2u}{dx^2}-y^2\frac{d^2u}{dy^2}-z^2\frac{d^2u}{dz^2}-2x\frac{du}{dx}+2y\frac{du}{dy}+2z\frac{du}{dz}.$$

Multiplying this by $3u$ or its equivalent, this becomes

$$2yz\left(3u\frac{d^2u}{dy\,dz}-2\frac{du}{dy}\frac{du}{dz}\right)$$
$$=x^2\left\{3u\frac{d^2u}{dx^2}-2\left(\frac{du}{dx}\right)^2\right\}-y^2\left\{3u\frac{d^2u}{dy^2}-2\left(\frac{du}{dy}\right)^2\right\}-z^2\left\{3u\frac{d^2u}{dz^2}-2\left(\frac{du}{dz}\right)^2\right\}$$

whence

$$2lmn\,\Sigma lyz\left(3u\frac{d^2u}{dy\,dz}-2\frac{du}{dy}\frac{du}{dz}\right)$$
$$=lmn\,\Sigma(l-m-n)x^2\left\{3u\frac{d^2u}{dx^2}-2\left(\frac{du}{dx}\right)^2\right\}.$$

Now $l^2x-lmx-lnx=l^2x+m^2y+n^2z+mn(y+z)$ from the first condition, and the right-hand side becomes

$$\Sigma m^2n^2x(y+z)\left\{3u\frac{d^2u}{dx^2}-\left(\frac{du}{dx}\right)^2\right\}$$

from the second.

16189. (T. STUART, M.A., D.Sc.)—Find a direct, non-tentative, and quick process for solving the equation $x^2 \equiv a \pmod{p}$ where a is a quadratic residue of p and p is a large prime. Exhibit your method by solving the equations

(*a*) $x^2 \equiv -537 \pmod{6311}$, (*b*) $x^2 \equiv 3582 \pmod{7177}$,
(*c*) $x^2 \equiv 3789 \pmod{8513}$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The following method of solving the congruence $x^2 \equiv a \pmod{p}$ (a given, x sought), though certainly tentative, seems worth publishing, as it has been constantly used with success by the writer.

Add mp to a, and factorize $(a+mp)$; so that

$$a \equiv a+mp \equiv (a_1a_2\ldots a_r)\,b^2C \pmod{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $a_1, a_2, \ldots, a_r$ are small (non-square) integers such that for each one a partition of type p or $m_r p = t_r^2 \pm a_r u_r^2$ .....(2) can be easily effected, and C any other integer (free from squares, all the squares being included in b^2). Hence arise r congruences of type

$$a_r u_r^2 \equiv \mp t_r^2 \pmod{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Multiplying together all the congruences (1), (3), all the non-squares $(a_1, a_2, \ldots, a_r)$ will cancel out, leaving

$$a\,(u_1 u_2 \ldots u_r)^2 \equiv \pm (b t_1 t_2 \ldots t_r)^2\, C \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

As a variation on above (3), any two of the (non-squares) a_r, say a_1, a_2, may be eliminated by a single partition of type

$$p \text{ or } m_1 p = a_1 t_2^2 \pm a_2 u_2^2, \text{ whence } a_1 a_2 u_2^2 \equiv \mp (a_1 t_2)^2 \pmod{p} \ldots (4').$$

Next, let C be treated in the same way as a was. If a new quantity C′ appears which is not readily removed by a partition of type (3), the process must be repeated until all the non-squares are eliminated (except a itself): this leaves a result of type

$$a U^2 \equiv + T^2 \text{ or } a \equiv (T/U)^2 \pmod{p}.$$

Hence $$a \equiv [(T \pm np)/U]^2 \equiv x^2 \pmod{p},$$

the multiplier being determined so as to render the fraction integral $(= x)$; then $\pm x$ are the required solutions.

Ex. (*a*).—$x^2 \equiv -537 \pmod{p = 6311}$. Here

$$-537 = -3 \cdot 179 \equiv -3 \cdot 6490 \equiv -30 \cdot 649$$
$$\equiv -30 \cdot 6960 \equiv -2 \cdot 29\,(60^2) \pmod{p} \ldots\ldots\ldots\ldots (A);$$

but $p = 29 \cdot 3^2 + 2 \cdot 55^2$, whence $29 \cdot 3^2 \equiv -2 \cdot 55^2 \pmod{p}$ ..... (B).

Multiplying (A) by (B) and cancelling gives $-537 \equiv (8 \cdot 5 \cdot 55)^2 \equiv 2200^2$ hence $x = \pm 2200$.

Ex. (*b*).—$x^2 \equiv 3582 \pmod{p = 7177}$. Here

$$3582 = 2 \cdot 3^2 \cdot 199 \equiv 2 \cdot 3^2 \cdot 7376 \equiv 2^5 \cdot 3^2 \cdot 461 \pmod{p}$$
$$\equiv 2^5 \cdot 3^2\,(-6716) \equiv -2^7 \cdot 3^2 \cdot 1679 \equiv -2^7 \cdot 3^2 \cdot 8856 \equiv -2^{10} \cdot 3^5 \cdot 41$$
$$\ldots\ldots\ldots\ldots (A);$$

but $p = 65^2 + 82 \cdot 6^2$, whence $2 \cdot 41 \cdot 6^2 \equiv -65^2 \pmod{p}$ .....(B).

(A) and (B) give $3582 \equiv 2^7 \cdot 3^3 \cdot 65^2 = 6\,(2^3 \cdot 3 \cdot 65)^2$ .... (C).

But $p = 109^2 - 6 \cdot 28^2$, whence $6 \cdot 28^2 \equiv 109^2 \pmod{p}$ ..... (D).

(C) and (D) give $3582 \cdot 28^2 \equiv (8 \cdot 3 \cdot 65 \cdot 109)^2 \equiv (170040)^2$.

Hence $$3582 \equiv \left(\frac{42510}{7}\right)^2 \equiv \left(\frac{42510 - 3p}{7}\right)^2 \equiv \left(\frac{20979}{7}\right)^2 = 2997.$$

and $$x = \pm 2997.$$

Ex. (*c*).—$x^2 \equiv 3789 \pmod{p = 8513}$. Here

$$3789 = 3^2 \cdot 421 \equiv 3^2\,(-8092) \equiv -7\,(2 \cdot 3 \cdot 17)^2 \ldots\ldots\ldots\ldots (A).$$

But $p = 55^2 + 7 \cdot 28^2$, whence $7 \cdot 28^2 = -55^2 \pmod{p}$ .....(B).

(A) and (B) give $3789 \cdot 28^2 \equiv (2 \cdot 3 \cdot 17 \cdot 55)^2 \equiv (2 \cdot 2805)^2$.

Hence $$3789 \equiv \left(\frac{2805}{14}\right)^2 \equiv \left(\frac{2805 - 5p}{14}\right)^2 \equiv \left(\frac{-39760}{14}\right)^2 \equiv (-2840)^2$$

and $$x = \pm 2840.$$

16143. (Professor NEUBERG.)—On considère les hyperboles équilatères qui passent par deux points fixes A, B et dont une asymptote passe par une point fixe C. Trouver (1) le lieu des centres de ces courbes; (2) l'enveloppe des secondes asymptotes.

Solutions (I.) *by* A. M. NESBITT, M.A.; (II.) *by* Professor SANJÁNA, M.A.

(I.) The rectangular hyperbola

$$x^2-y^2+xy(m-1/m)-(mb-c)(1+m^2)\,y/m^2=a^2$$

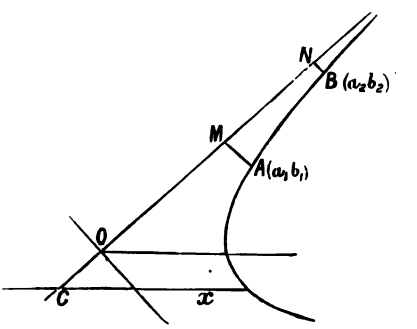
passes through the points $(\pm a, 0)$ and has for asymptotes $y-c=m(x-b)$ and $m^2y+m(x+b)=c$. Eliminating m, we get as the equation to the locus of centre the circular cubic

$$(y-2c)(x^2+y^2)+2bcx=y(b^2-c^2),$$

which passes through the origin (middle point of AB) and through C. The envelope of the second asymptote is easily seen to be the parabola $(x+b)^2=4cy$.

(II.) Let O be the centre;

$OC=r$, $\angle OCx=\theta$.



Then

$$OM.MA=ON.NB$$

gives

$$(CM-CO)MA=(CN-CO)NB,$$

i.e.,

$$(a_1\cos\theta+b_1\sin\theta-r)\times(a_1\sin\theta-b_1\cos\theta)=(a_2\cos\theta+b_2\sin\theta-r)\times(a_2\sin\theta-b_2\cos\theta)\ldots(1).$$

This is the polar equation of the locus; in rectangular co-ordinates we get

$$(x^2+y^2)(a_1y-b_1x-a_2y+b_2x)=(x^2-y^2)(a_2b_2-a_1b_1)+xy(a_1^2-b_1^2-a_2^2+b_2^2),$$

a cubic through C.

The equation of the other asymptote is $x\cos\theta+y\sin\theta=r$; and from (1) r may be found. Thus we get

$$(x\cos\theta+y\sin\theta)(a_1\sin\theta-b_1\cos\theta-a_2\sin\theta+b_2\cos\theta)=(a_1\cos\theta+b_1\sin\theta)(a_1\sin\theta-b_1\cos\theta)-(a_2\cos\theta+b_2\sin\theta)(a_2\sin\theta-b_2\cos\theta);$$

whence and from the difference equation eliminating θ we can get the envelope.

16216. (W. F. BEARD, M.A.)—Two similar ellipses with their axes parallel are cut by any straight line in A, B, A′, B′; a point O is taken in AD such that $OA.OB=OA'.OB'$. Prove that the locus of O is a straight line.

Solution by HENRY RIDDELL, M.E., *and others.*

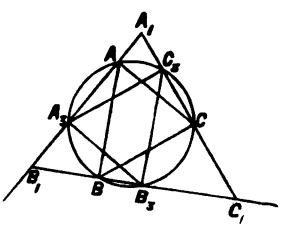
Both conics will project orthogonally at the same time into circles. The locus of O is now plainly the radical axis of the two circles, and, as the property $OA.OB = OA'.OB'$ is unchanged by orthogonal projection, the locus of O on reprojection is the projection of the radical axis.

This is also quite easily seen from the ordinary geometrical property of the ellipse—the rectangles under the segments of any chords passing through a point are proportional to the squares of the diameters parallel to the same. Hence at once the locus is the line which passes through the two points of intersection (real or imaginary) of the similar and similarly placed ellipses.

15874. (N. PARAMESWARA MENON.) — ABC is a given triangle; through A, A_1B_1 is drawn parallel to the perpendicular from B on AC; through B, B_1C_1 is drawn parallel to the perpendicular from C on AB; and, through C, C_1A_1 is drawn parallel to the perpendicular from A on BC. Find (i.) the area of the triangle $A_1B_1C_1$ in terms of that of the triangle ABC; (ii.) the circum-radius of the triangle $A_1B_1C_1$; (iii.) the area of the pedal triangle of $A_1B_1C_1$ in terms of that of the pedal triangle of ABC; and (iv.) show that, if A_1B_1 cut the circum-circle of ABC in A_2, and B_2, C_2 are the two other points obtained in the same way, the triangle $A_2B_2C_2$ is identically equal to the triangle ABC.

Solutions (I.) *by* R. J. WHITAKER; (II.) *by* M. V. ARUNACHALAN, M.A.

(I.) Let Δ, Δ_1 represent the areas of the triangles ABC, $A_1B_1C_1$; P, P_1 the areas of the pedal triangles; and R, R_1 the radii of the circum-circles. Now A_1B_1, B_1C_1 are perpendicular to AC, AB; therefore $B_1 = A$. Similarly $C_1 = B$ and $A_1 = C$.



(i.) $$\Delta_1 = 2R_1^2 \sin A \sin B \sin C$$
$$= 2\frac{R^4(a^2+b^2+c^2)^2}{a^2b^2c^2}\frac{abc}{8R^3}$$
$$= \frac{R(a^2+b^2+c^2)^2}{4abc} = \frac{(a^2+b^2+c^2)^2}{16\Delta}.$$

(ii.) Now
$$R_1 = \frac{a_1}{2\sin A_1} = \frac{c\cot A + a\operatorname{cosec} B}{2\sin C} = \frac{c\cos A \sin B + a\sin A}{2\sin A \sin B \sin C}$$
$$= \frac{a^2+bc\cos A}{4R\sin A\sin B\sin C} = \frac{a^2+b^2+c^2}{8R\sin A\sin B\sin C} = \frac{R^2(a^2+b^2+c^2)}{abc}.$$

(iii.) $P_1 = \frac{1}{2}R_1^2 \sin 2A_1 \sin 2B_1 \sin 2C_1$ and $P = \frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C$

therefore $$P_1 = \frac{R_1^2}{R^2}P = \frac{R^2(a^2+b^2+c^2)^2}{a^2b^2c^2}P = \left(\frac{a^2+b^2+c^2}{4\Delta}\right)P.$$

(iv.) Arc CA_3 = arc BC_3; therefore arc BC = arc A_3C_3; therefore

$$B_3 = A \quad \text{and} \quad b_3 = a, \ldots.$$

(II.) It is easily shown that the angles at A_1, B_1, C_1 are respectively equal to C, A, B in the original triangle. Let Ω' be the negative B.-point of the triangle ABC. Then

$$\angle A\Omega'B = \pi - A = \pi - B_1.$$

Hence the figure $AB_1B\Omega'$ is cyclic; therefore

$$\angle A\Omega'B = \angle B_1BA = \text{a right angle.}$$

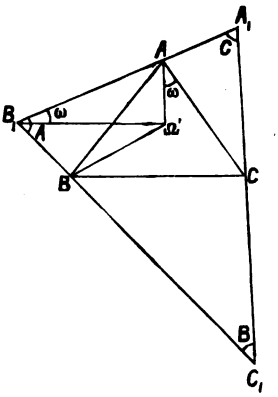
Similarly,

$$\angle B\Omega'C_1 = \angle A_1\Omega'C = \text{a right angle;}$$

therefore $\angle A\Omega'B = \angle B_1\Omega'C_1$;

therefore $\angle B_1\Omega'C_1 = \pi - A = \pi - B_1$.

Hence it follows that Ω' is the negative B.-point of $A_1B_1C_1$; therefore the triangles ABC, $A_1B_1C_1$ are similar, their centre of similitude being Ω', their common negative B.-point.



(i.) Therefore

$$\triangle A_1B_1C_1 : \triangle ABC = \Omega'B_1 : \Omega'A = \sin B_1A\Omega' : \sin AB_1\Omega' = \cos\omega : \sin\omega.$$

(ii.) Similarly, the circum-radius of the triangle $A_1B_1C_1$ = cot ω multiplied by the circum-radius of the triangle ABC.

(iii.) Similarly, the area of the pedal triangle of the triangle $A_1B_1C_1$ = cot ω multiplied by that of the pedal triangle of ABC.

(iv.) If the circum-circle of ABC cut the sides of $A_1B_1C_1$ in A_3, B_3, C_3, the triangle $A_3B_3C_3$ is evidently equiangular with the triangle ABC, and, being inscribed in the same circle, the two triangles are identically equal.

The PROPOSER'S Solution is as follows:—

(i.)

$$A_1B_1 = b\cot C + c\operatorname{cosec} A$$

$$= \frac{2R\sin B(1+\pi\cos A)}{\pi\sin A} \quad \ldots\ldots\ldots (\alpha).$$

Similarly,

$$B_1C_1 = \frac{2R\sin C(1+\pi\cos A)}{\pi\sin A} \quad \ldots\ldots\ldots (\beta).$$

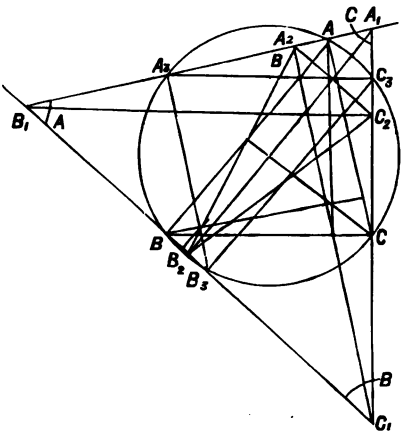
Therefore

area of triangle $A_1B_1C_1$

$$= \tfrac{1}{2}\frac{4R^2\sin B\sin C}{\pi\sin^2 A}\times(1+\pi\cos A)^2\sin A$$

$$= \frac{2R^2(1+\pi\cos A)^2}{\pi\sin A}$$

$$= \Delta_1 \text{ (say).}$$



If $\Delta = \frac{1}{2}(bc \sin A) = 2R^2 \sin A \sin B \sin C$,

then $\Delta_1^{\frac{1}{2}} = (\pi \operatorname{cosec} A + \pi \cot A)\, \Delta^{\frac{1}{2}}$.

(ii.) If R_1 be the circum-radius, we have

$$[2R \sin B (1 + \pi \cos A)]/(\sin B\, \pi \sin A) = 2R_1 ;$$

therefore $R_1 = R(\pi \operatorname{cosec} A + \pi \cot A)$.

(iii.) $A_2B_1 = [2R \sin C (1 + \pi \cos A)] \cos A/\pi \sin A$;

therefore $A_2B_2 = [2R(1 + \pi \cos A) \cos A]/\sin B \sin C$.

Similarly, $B_2C_2 = [2R \cos B (1 + \pi \cos A)]/\sin A \sin C$;

therefore area of $A_2B_2C_2 = \dfrac{4R^2 \cos A \cos B (1 + \pi \cos A)^2 \sin 2C}{\pi \sin A \sin C}$;

i.e., Δ_2 (say) $= [8R^2 \pi \cos A (1 + \pi \cos A)^2]/\pi \sin A$.

But the area of the pedal triangle of ABC can be shown to be equal to $4R^2 \pi \sin A\, \pi \cos A = \Delta_{11}$ (say) ; therefore

$$\Delta_2^{\frac{1}{2}} = \sqrt{2}\,(\pi \operatorname{cosec} A + \pi \cot A)\, \Delta_{11}^{\frac{1}{2}}.$$

(iv.) From the figure we see that $\angle AA_2C_2 = \frac{1}{2}\pi - \angle C$; therefore $\angle B_2A_2C_2 = \angle C$. Similarly, $\angle A_2B_2C_2 = \angle A$ and $\angle A_2C_2B_2 = \angle B$. Also $\angle A_2AC_2 = \pi - A$; therefore $A_2C_2 = 2R \sin A$. But $a = 2R \sin A$; therefore $A_2C_2 = a$. Similarly, the other two sides are equal to b and c. Thus

$$\Delta A_2B_2C_2 \equiv \Delta ABC.$$

16188. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Determine the sides of two right-angled triangles in integers so that the product of the hypothenuse and a side of one may be equal to the product of the hypothenuse and a side of the other.

Remarks by Professor SANJÁNA, M.A.

Proceeding as in Question 15934 (Mr. A. Holm, M.A.), the sides of the triangles may be taken as $l_1(m_1^2 + n_1^2)$, $2l_1m_1n_1$, $l_1(m_1^2 - n_1^2)$; $l_2(m_2^2 + n_2^2)$, $2l_2m_2n_2$, $l_2(m_2^2 - n_2^2)$. If the products of the first two are equal, then, assuming $l_1^2 = m_2^2 + n_2^2$, $l_2^2 = m_1^2 + n_1^2$, we have

$$m_1n_1 = m_2n_2.$$

Hence $l_1 = \lambda_2(\mu_2^2 + \nu_2^2)$, $m_2 = 2\lambda_2\mu_2\nu_2$, $n_2 = \lambda_2(\mu_2^2 - \nu_2^2)$;

$l_2 = \lambda_1(\mu_1^2 + \nu_1^2)$, $m_1 = 2\lambda_1\mu_1\nu_1$, $n_1 = \lambda_1(\mu_1^2 - \nu_1^2)$;

and thus, finally, $\lambda_1^2\mu_1\nu_1(\mu_1^2 - \nu_1^2) = \lambda_2^2\mu_2\nu_2(\mu_2^2 - \nu_2^2)$;

so that $[\mu_1\nu_1(\mu_1^2 - \nu_1^2)]/[\mu_2\nu_2(\mu_2^2 - \nu_2^2)] = \lambda_2^2/\lambda_1^2$.

We may then assume

$$\mu_1 = 2\lambda_2^2 - \lambda_1^2, \quad \nu_1 = \lambda_2^2 + \lambda_1^2, \quad \mu_2 = \lambda_2^2 + \lambda_1^2, \quad \nu_2 = \lambda_2^2 - 2\lambda_1^2.$$

As Mr. Holm correctly says, the method does not furnish all solutions ; and the smaller solutions are very difficult to obtain : *e.g.*, even when $\lambda_2 = 2$, $\lambda_1 = 1$, the triangles obtained are 29.37^2, 29.840, 29.1081 ;

37.29^2, 37.840, 37.41. The following method is practical only, though some parts of it can be easily generalized :—

I. Let one hypothenuse be a square number $N = P^2 = c^2 + d^2$; the sides are $2c$, c^2-d^2. Let either of these $= mn$ (or $m'n'\delta^2$); then, if m^2-n^2 (or $m'^2\delta^2-n'^2\delta^2$) $= \square = y^2$, m and n are the hypothenuse and one side of a right-angled triangle. The original triangle and the triangle (mP, nP, yP) form the solution. *Exs.:* 289, 240, 161; $240 = 20.12 = 16^2-4^2 = 4^4-2^4$; hence 4^2+2^2, 4^2-2^2, 2.4.2 is the second triangle, and, enlarging, we get 340, 204, 272. Similarly, from 1681, 720, 1519 we get $720 = 36\,(5.4)$, so that $30^2-24^2 = 18^2$; hence 30.41, 24.41, 18.41 is the second triangle.

II. Let one hypothenuse be $P^2+Q^2 = l^2(c^2+d^2)$; let 2PQ or P^2-Q^2 be a perfect square δ^2; then, if c^2-d^2 or $2cd$ is also a perfect square ϵ^2, the original triangle enlarged δ times and the triangle (c^2+d^2, $2cd$, c^2-d^2) enlarged $l\epsilon$ times will furnish solutions. Thus, from 1025, 64, 1023 and 41, 40, 9 we get 3075, 192, 3069 and 1640, 360, 1600. If 2PQ or P^2-Q^2 be of the form $m\delta^2$, and m, c^2+d^2 form the hypothenuse and one side, the triangles can be similarly obtained. Thus, from $125\,(=5.5^2)$, $117\,(=13.3^2)$, 44, we get 5, 13, 12, and, enlarging, 195, 75, 180 is the second triangle. So also from 1625, 1184, 1113 we can deduce 1850, 1040, 1530.

III. When the hypothenuse does not contain a square factor it must be of the form $P^2+Q^2 = (a^2+b^2)(c^2+d^2)$. In this case let P^2-Q^2 or 2PQ be of the form $m\delta^2$. If from m, a^2+b^2, c^2+d^2 or these multiplied by square factors we can get a right-angled triangle, the Question is solved. Thus, from 221 (13.17), 171 (9.19), 140, we get 13.35, 19.17, which with 36 form a right-angled triangle; hence the following solution is obtained :—

1105, 855, 700; 975, 969, 108.

The following Solution is due to the Proposer :—

Let (a, b, c), (A, B, C) be the sides of the two triangles, so that

$$c^2 = a^2+b^2, \quad C^2 = A^2+B^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The general solution of these in integers are known to be expressible in terms of the arbitrary integers (t, u), (T, U)

$$a = t^2-u^2, \quad b = 2tu, \quad c = t^2+u^2 \quad \ldots\ldots\ldots\ldots (2a);$$

$$A = T^2-U^2, \quad B = 2TU, \quad C = T^2+U^2 \quad \ldots\ldots\ldots\ldots (2b).$$

And the given relation between the triangle requires either

$$\text{(i.)}\ bc = BC, \quad \text{(ii.)}\ ac = AC \quad \ldots\ldots\ldots\ldots (3).$$

Case i.—$bc = BC$ gives $tu(t^2+u^2) = TU(T^2+U^2)$ ..... (4).

Euler has given (*Op. Arithm.*, p. 289, *et seq.*) the general solution of the latter in terms of two new arbitraries (f, g),

$$t = g(f^2+g^2)(-f^4+18f^2g^2-g^4), \quad u = 2f(f^6+10f^4g^2+f^4g^2+4g^6) \ldots (5a),$$
$$T = 2g(4f^6+f^4g^2+10f^2g^4+g^6), \quad U = f(f^2+g^2)(-f^4+18f^2g^2-g^4) \ldots (5b).$$

Even small values of f, g yield high values of t, u, T, U; but in many cases these contain a common factor which may be removed, giving

effectively lower values of t, u, T, U. Euler shows that the solutions exist in conjugate pairs, such that, if (t, u, T, U) be a solution, then (t', u', T', U'), given by

$$t' = (t-u)+(T-U), \quad u' = (t-u) \sim (T-U),$$
$$T' = (t+u)+(T+U), \quad U' = (t+u)+(T+U),$$

are also a solution. Hence the triangle solutions also exist in conjugate pairs. Euler's lowest solutions are the conjugates

$(t, u; T, U) = (146, 12; 96, 37)$; $(t', u'; T', U') = (291, 25; 193, 75)$.

These lead to the conjugate pairs of triangles by equations (2a), (2b),

$$\left.\begin{aligned}(a, b, c) &= (7.19.59,\ 64.3.37,\ 5.29.73)\\ (A, B, C) &= (4.59.79,\ 16.3.73,\ 4.5.29.37)\end{aligned}\right\},$$
$$\left.\begin{aligned}(a', b', c') &= (3.59.67,\ 2.3.25.193,\ 2.3.17.97)\\ (A', B', C') &= (3.7.19.79,\ 2.3.25.97,\ c = 2.13.17.1937)\end{aligned}\right\}.$$

The common factor 2 may be cancelled from the latter set. The next solutions are of course in much higher numbers.

Case ii.—$ac = AC$ gives $t^4 \cdot u^4 = T^4 - U^4$ ..... (6).

Euler has given (*loc. cit.*) solutions for this form also, and Desboves has also given solutions in a general form [but longer than equations (5a), (5b) above], in his paper "Sur la résolution en nombres entiers ou complexes de l'équation $U^n \pm V^n = S^n + W_n$," Paris, 1880; but, as they lead to the same triangles as in Case i., this case need not be developed.

16197. (L. Isserlis, B.A.)—A straight line moves so as to cut three fixed conics in six points forming an involution. Find the locus of the centre of the involution.

Solution by S. T. Shovelton.

Let the three conics be

$$\phi_1(x, y) \equiv a_1x^2 + 2h_1xy + b_1y^2 + 2f_1y + 2g_1x + c_1 = 0,$$
$$\phi_2(x, y) = 0,$$
$$\phi_3(x, y) = 0,$$

and let (α, β) be the centre of the involution formed on the line

$$(x-\alpha)/l = (y-\beta)/m.$$

If $(\alpha + lr, \beta + mr)$ is on the first conic, we have

$$r_1r_2 = [\phi_1(\alpha, \beta)]/(a_1l^2 + 2h_1lm + b_1m^2),$$

which is therefore equal to

$$[\phi_2(\alpha, \beta)]/(a_2l^2 + 2h_2lm + b_2m^2) = [\phi_3(\alpha, \beta)]/(a_3l^2 + 2h_3lm + b_3m^2).$$

Let $\Delta = \begin{vmatrix} a_1 & h_1 & b_1 \\ a_2 & h_2 & b_2 \\ a_3 & h_3 & b_3 \end{vmatrix}$, and let the first minors be denoted by $A_1, H_1, B_1, \ldots$. Then the fractions are equal to

$$\frac{A_1\phi_1 + A_2\phi_2 + A_3\phi_3}{\Delta l^2} = \frac{H_1\phi_1 + H_2\phi_2 + H_3\phi_3}{2\Delta lm} = \frac{B_1\phi_1 + B_2\phi_2 + B_3\phi_3}{\Delta m^2};$$

therefore the locus is

$$(H_1\phi_1 + H_2\phi_2 \cdot H_3\phi_3)^2 = 4\,(A_1\phi_1 + A_2\phi_2 + A_3\phi_3)\,(B_1\phi_1 + B_2\phi_2 + B_3\phi_3),$$

which represents a cubic curve, since the terms of the second degree in the three brackets are $2\Delta xy$, Δx^2, Δy^2 respectively.

16118. (T. STUART, D.Sc.)—Solve in positive integers the equation $x^3 + y^3 = 53z^3$. Is the equation $x^3 + y^3 = 66z^3$ possible in integers?

Partial Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The two questions may be conveniently studied in the general forms

$$x^3 + y^3 = Cz^3, \quad x^3 + y^3 = 3Cz^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1, 2).$$

Write, for shortness,

$$x^3 + y^3 = X.Y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$x + y = X, \quad x^2 - xy + y^2 = Y, \quad z = \zeta Z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

It is assumed that x, y contain no factor of C or z: for any such common factor might be divided out, leaving a reduced equation of the original form (1) or (2).

The form of C in (1), (2) may also be limited to suit the present questions: thus

$$C \not\equiv 3c \not\equiv 6\gamma + 1 \text{ and contains no cube} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

It is known that X, Y can contain no common factor but 3: also, if they contain 3, then Y contains a simple 3 (not 9) and X contains the rest of the powers of 3 entering into z^3 or $3z^3$ in (1), (2) respectively. There will now be three cases requiring separate consideration.

Case i.—$XY = C\zeta^3 Z^3$, with X prime to Y. This involves

$$X = C\zeta^3, \quad Y = Z^3 \quad [Z \not\equiv 3Z'] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Case ii.—$XY = C\zeta^3 Z^3$, with $\zeta = 3\zeta'$. This involves

$$X = 9C\zeta'^3, \quad Y = 3Z^3 \quad [Z \not\equiv 3Z'] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Case iii.—$XY = 3C\zeta^3 Z^3$, where $\zeta = 3\zeta'$. This involves

$$X = C\,(3\zeta')^3, \quad Y = 3Z^3 \quad [Z \not\equiv 3Z'] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now, since Z^3 is a factor of Y, and $Z \not\equiv 3Z'$, Z must be of form

$$Z = A^2 + 3B^2, \text{ with } A \not\equiv 3a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

whence

$$Z^3 = A'^2 + 3B'^2, \text{ with } A' \not\equiv 3a' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9a),$$

where

$$A' = \pm A\,(A^2 - 9B^2), \quad B' = 3B\,(A^2 - B^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Here, if A, B be both odd, then A', B' both $\equiv 0 \pmod 8$, and Z would $\equiv 0 \pmod 4$, $Y \equiv 0 \pmod{64}$.

Also Y is expressible in the same 2-ic form as Z, Z^3, thus

$$Y = (x \sim \tfrac{1}{2}y)^2 + 3\,(\tfrac{1}{2}y)^2 \quad [\text{when } y \text{ is even}] \quad \ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

$$= \left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x \sim y}{2}\right) \quad [\text{when } xy \text{ is odd}] \quad \ldots\ldots\ldots\ldots (12b).$$

These 2-ic forms of Z^3, Y in equations (9a), (12a), (12b) may be identified in each of Cases i., ii., iii. through equations (6), (7), (8), giving in each

case two equations by which x, y may be expressed in terms of A, B. And x, y may be eliminated by the first or X-equation of each of equations (6), (7), (8), leaving in each case a relation between A, B, C, ζ. The sub-cases when y is even, or y is odd, require separate treatment, and the double signs in A', B' must also be considered.

Cases i., ii., iii. *with y even.*—Taking first the case of y even, from equation (12*a*),

Case i. (y even).—$Y = Z^3$ gives with equations (9*a*), (12*a*)

$$x - \tfrac{1}{2}y = \pm A', \quad \tfrac{1}{2}y = \pm B', \quad x = \pm A' \pm B' \quad \ldots\ldots\ldots\ldots (13),$$

whence $$X = x + y = \pm A' \pm 3B' = C\zeta^3 \ldots\ldots\ldots\ldots (13a).$$

Case ii. (y even).—$Y = 3Z^3$ gives with equations (9*a*), (12*a*)

$$x - \tfrac{1}{2}y = \pm 3B', \quad \tfrac{1}{2}y = \pm A', \quad x = \pm A' \pm 3B' \quad \ldots\ldots\ldots\ldots (14),$$

$$X = x + y = \pm 3A' \pm 3B' = 9C\zeta'^3 \ldots\ldots\ldots\ldots (14a).$$

Case iii. (y even).—$Y = 3Z^3$ gives with equations (9*a*), (12*a*)

$$x - \tfrac{1}{2}y = \pm 3B', \quad \tfrac{1}{2}y = \pm A', \quad x = \pm A' \pm 3B' \quad \ldots\ldots\ldots\ldots (15),$$

$$X = x + y = \pm 3A' \pm 3B' = 27C\zeta'^3 \ldots\ldots\ldots\ldots (15a).$$

Note that the condition y even in each of the above Cases i., ii., iii. involves x odd (since x is prime to y). The formulæ (13), (14), (15) for x now involve one of A', B' odd and one even; and the formulæ (10) for A', B' then involve one of A, B odd, and one even; hence Z must be odd.

As to Cases ii., iii. above, equations (14*a*), (15*a*) alike require $\pm A' \pm B' \equiv 0 \pmod 3$. But $B' \equiv 0 \pmod 3$ by (10), and thus involves $A' \equiv 0 \pmod 3$, thereby violating result (9*a*). Both Cases ii., iii. are thus impossible (with y even).

Returning to Case i. (y even), equations (13*a*) and (10) give

$$\pm A(A^2 - 9B^2) \pm 9B(A^2 - B^2) = C\zeta^3 \ldots\ldots\ldots\ldots (16).$$

There appears to be no known general way of solving any one of these four equations (16) in A, B, ζ for a given value of C. But it is clear that every value of C, limited as in (5), yielding solvable equations (1), (2), with y even, must be of one of the four forms (16), and must therefore be odd (and ζ must be odd). And, by trial with small numerical values of A, B, values of C will be found leading to solvable equations: thus the values of A, B below (with proper signs attached) lead to the values of C shown

$$(A, B) = (2, 1), \ (5, 8), \ (1, 2), \ (1, 5), \ \ldots,$$
$$C = 17, \quad 53, \quad 89, \quad 107, \quad \ldots.$$

Ex.—A = 5, B = 8 give

$$A' = A(9B^2 - A^2) = 2755, \quad B' = 3B(B^2 - A^2) = 936;$$

whence $$x = B' - A' = -1819, \quad y = 2B' = 1872,$$

giving $x + y = 53 = C\zeta^3$; C = 53, $\zeta = 1$; $Z = A^2 + 3B^2 = 217$.

The direct solution for C = 53 in positive integers would appear very laborious; but M. Desboves gives, in his *Mémoire sur la résolution en nombres entiers de l'équation* $aX^m + bY^m = cZ^m$ (1879), formulæ whereby any number of solutions may be formed in succession from a single (known) solution; some of these will probably be in positive

integers (but these do not seem to lead to positive solutions in the case of C = 53).

Cases i., ii., iii., *with y odd.*—Next, taking up the case of y odd from equation (12*b*),

Case i. (y odd.)—$Y = Z^3$ gives with equations (9*a*), (12*b*)

$\frac{1}{2}(x+y) = \pm A'$, $\frac{1}{2}(x-y) = \pm B'$; $x = \pm A' \pm B'$, $y = \pm A' \mp B'$...(17),

whence $X = x+y = \pm 2A' = C\zeta^3$ .....(17*a*).

Case ii. (y odd).—$Y = 3Z^3$ gives with equations (9*a*), (12*b*)

$\frac{1}{2}(x+y) = \pm 3B'$, $\frac{1}{2}(x-y) = \pm A'$; $x = \pm A' \pm 3B'$, $y = \pm 3B' \mp A'$...(18),

whence $X = x+y = \pm 6B' = 9C\zeta'^3$ .....(18*a*).

Case iii. (y odd).—$Y = 3Z^3$ gives with equations (9*a*), (12*b*)

$\frac{1}{2}(x+y) = \pm 3B'$, $\frac{1}{2}(x-y) = \pm A'$; $x = \pm A' \pm 3B'$, $y = \pm 3B' \mp A'$...(19),

whence $X = x+y = \pm 6B' = 27C\zeta'^3$ .....(19*a*).

Note that the condition x, y both odd in each of the above i., ii., iii. involves one of A', B' odd, and one even; and the formulæ (10) for A', B' then involve one of A, B odd, one even; hence Z must be odd, and therefore also Y is odd, and X even in each case.

The three results (17*a*), (18*a*), (19*a*) now reduce as follows:—

Case i.— $\pm 2A(A^2-9B^2) = C\zeta^3$ .....(20).

Case ii.— $\pm 2B(A^2-B^2) = C\zeta'^3$ .....(21).

Case iii.— $\pm 2B(A^2-B^2) = 3C\zeta'^3$ .....(22).

And, since A should be prime to 3B, and A, B should be one odd, one even, these formulæ involve that the three symbolic factors $A(A \pm 3B)$ in Case i. and $B(A \pm B)$ in Case ii. should themselves be of the forms $C_1\zeta_1^3$, $C_2\zeta_2^3$, $C_3\zeta_3^2$.

Writing $A+3B = m$, $A-3B = n$ in (20), or $A+B = n$, $A-B = m$ in (21), (22), these equations may be reduced to the type

$$mn(m+n) = C\zeta^3 \text{ or } 3C\zeta'^3 \quad \text{.....(23).}$$

If the three factors of (20), (21), (22) be written in the forms $C_1\zeta_1^3$, $C_2\zeta_2^3$, $C_3\zeta_3^3$ above explained, the elimination of A, B will lead in each case to an equation of the type

$$\lambda t^3 + \mu u^3 = \nu w^3 \quad \text{.....(24),}$$

the solution of which would carry with it that of the original (20), (21), (22); but this seems at least as difficult as the original. There seems to be no known general way of solving any of the equations (20), (21), (22), or the allied forms (23), (24) for a given value of C. But it is clear that every value of C, limited as in (5), which yields solvable equations (1), (2) with xy odd must be of one of the six forms (20), (21), (22), and therefore of the form (23). Even values of C require xy odd (for it was shown above that y even gives only odd values of C), and therefore always fall under forms (20), (21), (22). And, by trial with small numerical values of A, B, values of C will be found leading to solvable equations (1), (2).

The further discussion will now, for brevity's sake, be limited to Case iii., equation (22), with C further limited to

$$C = 2q, \text{ where } q = 6\kappa - 1, \text{ a prime} \quad \text{.....(25),}$$

as the proposed case of 3C = 66 falls under this form. The condition (22) here reduces to

$$\pm B(A^2 - B^2) = 3q\zeta'^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

And here B, (A ~ B), (A + B) must be of following forms :—

$$\begin{array}{rcccc} B = 3q\beta^3 & 3\beta^3 & q\beta^3 & \beta^3 & \\ A \sim B = t^3 & qt^3,\ t^3 & 3t^3,\ t^3 & 3qt^3,\ qt^3,\ 3t^3,\ t^3 & \\ A + B = u^3 & u^3,\ qu^3 & u^3,\ 3u^3 & u^3,\ 3u^3,\ qu^3,\ 3qu^3 & \end{array} \quad \ldots\ldots (27).$$

Eliminating A, B leads to equations of following types only (t, u being here interchangeable) :—

$$\left.\begin{array}{c} u^3 \pm t^3 = 6q\beta^3, \quad u^3 \pm qt^3 = 6\beta^3, \quad u^3 \pm 3t^3 = 2q\beta^3 \\ u^3 \pm 3qt^3 = 2\beta^3, \quad 3u^3 \pm qt^3 = 2\beta^3 \end{array}\right\} \quad \ldots\ldots (28).$$

This is the system of "allied equations" (24) mentioned above, the solution of any one of which carries with it that of the original (26); but these equations appear as difficult of solution as the original (22).

Reverting, however, to the condition (26) and removing the condition that q should be prime, values of q leading to solvable equations of type (2) may be formed by assigning small numerical values to A, B. Noting that the condition $q = 6\kappa - 1$ involves with what precedes either $B = 3b$, $A = 3a \pm 1$; or $A = 3a \pm 1$, $B = 3b \pm 1$, and A, B should be one odd, one even, then

$$A = 2, 4, 13; \quad B = 1, 1, 8; \quad 3q\zeta'^3 = 3, 15, 105 \cdot 2^3.$$

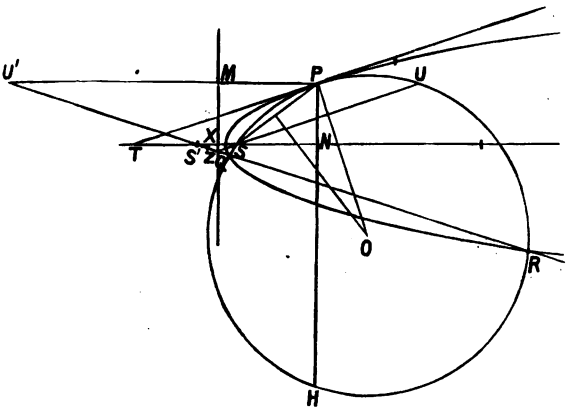
But the value $q = 11$ (which gives 3C = 66) does not appear to result for any small values of A, B; it would seem very difficult to prove that 3C = 66 was an impossible case.

16123. (W. F. Beard, M.A.) — If a circle be drawn through the focus of a parabola to touch it at P and cut it at Q, R, then QR passes through a fixed point.

Solutions (I.) *by* James Blaikie, M.A.; (II.) *by* Professor Nanson; (III.) *by* A. M. Nesbitt, M.A.

(I.) This problem, with others, is solved in Milne and Davis, *Geometrical Conics*, pp. 50, 51. The following proof is in its main points the same as that given there. Let O be the centre of the circle and let QR meet the axis in S′ and the diameter through P in U′. Also let U be the point where the diameter through P meets the circle. Then, since the normal PO makes equal angles with SP and PU, the chord SP = PU = PM, and $\angle PSU = \angle SUP = \frac{1}{2}\angle SPM = \angle SPT$, or SU is parallel to TP. Also, since the secant QR meets the diameter through P in U′, we have $U'Q.U'R = 4SP.PU'$ (Milne and Davis, xxv.). But, by the property of the circle, $U'Q.U'R = U'P.U'U$; therefore $4SP = U'U = 4PU = 2UM$; therefore $U'M = UM$, or U′ is the image of U in the directrix. And we know that when a parabola

and a circle intersect their common chords make equal angles with the directrix; therefore QR is equally inclined to the directrix with TP



and therefore with SU. It follows that U'S' is the image of US in the directrix, and S' is the image of S in the directrix, and is therefore a fixed point.

(II.) It is sufficient to prove that, if an in-conic and a circum-conic of a triangle touch, then the common chord of the conics goes through a fixed point. Now any in-conic is

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and any conic touching (1) at x', y', z' and having (l, m, n) for common chord with (1) is

$$(x/\sqrt{x'} + y/\sqrt{y'} + z/\sqrt{z'})(lx + my + nz) = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy.$$

If this is a circum-conic, we must have $l = \sqrt{x'}$, $m = \sqrt{y'}$, $n = \sqrt{z'}$. Hence $l + m + n = 0$, and therefore the common chord goes through the point (1, 1, 1).

(III.) Any conic touching $y^2 = 4ax$ at $(a/l^2, 2a/l)$ and cutting it in $(am^2, 2am)$, $(an^2, 2an)$ may be written

$$\theta(y^2 - 4ax) - (y - lx - a/l)[(m + n)y - 2x - 2amn] = 0.$$

This is a circle if $\theta + m + n = 2l$, $2 + l(m + n) = 0$, and it passes through $(a, 0)$ if $2\theta = (l + 1/l)(1 + mn)$. From these we get at once $mn = 3$; so that the line $(m + n)y = 2x + 2amn$ goes through the fixed point $(-3a, 0)$.

The following Solution is due to the PROPOSER:—

With the figure of Solution (I.), let QR meet the directrix at Z and the axis at K. Let the tangent at P meet the axis at T. Produce the ordinate NP through N to meet the circle again at H. Join SH, SZ.

PQ, PR are equally inclined to the axis; therefore PN bisects the angle QPR; therefore arc QH = arc HR; therefore HS bisects the angle

QSR internally and ZS bisects it externally; therefore the angle ZSH is a right angle; therefore

$$\angle XSZ = 90° - \angle NSH = \angle SHN = \angle SPT,$$

since TP touches the circle,

$$= \angle ZKS,$$

since QR and PT are equally inclined to the axis.

Thus $\triangle XZS \equiv \triangle XZS'$ (Euc. I. 26); therefore $S'X = SX$; therefore S' is a fixed point.

16152. (M. V. ARUNACHALAN, M.A.)—Show that

$$[a(b-c)(bc-aa')(a^4-a'^4)]/(a-a') + [b(c-a)(ca-bb')(b^4-b'^4)]/(b-b') + [c(a-b)(ab-cc')(c^4-c'^4)]/(c-c')$$
$$= (b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,S/abc,$$

where $aa' = bb' = cc'$ and $S = a+b+c+a'+b'+c'$.

Another Solution by S. NARAYANA AIYAR.

This is only a particular case of the following more general question:—If S_{m-3} denote the sum of the $(m-3)$-ary products of the six quantities a, b, c, a_1, b_1, c_1, establish the above relation where instead of $a^4, a_1^4, \ldots$, we have $a^m, a_1^m, \ldots$ and S_{m-3} instead of S.

If ${}_nK_r$ denote the r-ary products of the n quantities $a_1, a_2, \ldots, a_n$, we have

$${}_nK_r = \Sigma \frac{a_s^{n+r-1}}{(a_s-a_1)(a_s-a_2)\ldots(a_s-a_n)}.$$

When $n = 6$ and the quantities are a, b, c, a_1, b_1, c_1, and $n = m-3$, we get

$$S_{m-3} = \Sigma \frac{a^{m+2}}{(a-b)(a-c)(a-a_1)(a-b_1)(a-c_1)}$$
$$= \frac{abc}{(a-b)(a-c)(b-c)} \Sigma \frac{a(b-c)a^m}{(a-a_1)(ab-bb_1)(ac-cc_1)}$$
$$= \frac{abc}{(a-b)(a-c)(b-c)(bc-aa_1)(ac-bb_1)(ab-cc_1)} \Sigma \frac{a(b-c)(bc-aa_1)a^m}{a-a_1},$$

since $aa_1 = bb_1 = cc_1$.

Taking the six terms corresponding to the six quantities a, b, c, a_1, b_1, c_1, we have

$$S_{m-3} = \frac{abc}{(a-b)(a-c)(b-c)(ab-cc_1)(ac-bb_1)(bc-aa_1)} \times \left[\frac{a(b-c)(bc-aa_1)(a^m-a_1^m)}{a-a_1} + \frac{b(c-a)(ca-bb_1)(b^m-b_1^m)}{b-b_1} + \frac{c(a-b)(ab-cc_1)(c^m-c_1^m)}{c-c_1}\right].$$

Therefore the expression within the rectangular brackets

$$= \frac{S_{m-3}}{abc}(a-b)(a-c)(b-c)(ab-cc_1)(ac-bb_1)(bc-aa_1).$$

Here, if we put $m = 4$, we have $S_{m-3} = a+b+c+a_1+b_1+c_1$, as given in the question.

16216. (W. F. Beard, M.A.)—Two similar ellipses with their axes parallel are cut by any straight line in A, B, A′, B′; a point O is taken in the straight line such that OA.OB = OA′.OB′. Prove that the locus of O is a straight line.

Analytical Solution by R. J. Whitaker, B.A., *and* A. M. Nesbitt, M.A.

Let the equations to the ellipses be

$$ax^2 + by^2 = 1 \quad \text{and} \quad a(x+h)^2 + b(y+k)^2 = c.$$

Let O be the point (x', y'), and the equation of AB

$$(x-x')/\cos\theta = (y-y')/\sin\theta = r.$$

Then OA, OB are the roots of the equation in r

$$a(r\cos\theta + x')^2 + b(r\sin\theta + y')^2 = 1,$$

whence $\quad \mathrm{OA.OB} = (ax'^2 + by'^2 - 1)/(a\cos^2\theta + b\sin^2\theta)$;

and, similarly,

$$\mathrm{OA'.OB'} = [a(x'+h)^2 + b(y'+k)^2 - c]/(a\cos^2\theta + b\sin^2\theta).$$

Therefore the locus of O is

$$ax^2 + by^2 - 1 = a(x+h)^2 + b(y+k)^2 - c,$$

i.e.,

$$2ahx + 2bky + ah^2 + bk^2 - c + 1 = 0,$$

a straight line. This straight line passes through the real points of intersection (if any) of the ellipses. Also, if $\mathrm{OA.OB} = m.\mathrm{OA'.OB'}$, m being constant, the locus of O is an ellipse similar to the given ellipses and passing through their points of intersection.

The above proof is applicable to similar hyperbolas, and a similar line of reasoning will establish the proposition in the case of two parabolas whose axes are parallel.

A Note on Freedom and Geometry.

By S. C. Bagchi, B.A., LL.B.

A plane may be considered as an aggregate of points (Cartesian view) or an aggregate of straight lines (Plückerian view). Both these views are united in the conception of *freedom*. The degree of freedom of a *variety* (*variété* of Laurent: see *Géom. gén.*) is the number of independent parameters required to specify the variety uniquely: *e.g.*, a point or a straight line has two degrees of freedom in two-dimensional space. A plane is therefore engendered by a variety with two degrees of freedom. If we diminish the freedom of such a variety by unity, we can make it obey a fixed law. This law is the equation to a curve; which is evident from the fact that $f(x, y, z) = 0$ (where x, y are the Cartesian co-ordinates of a point and $z = 1$ is a linear unit introduced to make f homogeneous) can always be found whenever either

(i.) $x = \phi_1(\lambda, \mu)$, $y = \phi_2(\lambda, \mu)$, $\mu = \psi(\lambda)$ hold simultaneously

or (ii.) $f_1(x, y, z, \alpha\beta) = 0$ is a linear equation where $\beta = \phi(\alpha)$ such that $f_1 = 0$ and $\partial f_1/\partial\alpha + \partial f_1/\partial\beta\, d\beta/d\alpha = 0$. In the latter case $f = 0$ is

an envelope of $f_1 = 0$, and, if f_1 is of the form $\phi_1 + \theta\phi_2$ where θ is a function of α and β, the curve $f = 0$ is reduced to a fixed point: viz., the intersection of $\phi_1 = 0$ and $\phi_2 = 0$. Therefore a variety with residual freedom 1 and of original freedom 2 always describes a plane curve.

Now let $S = 0$ and $S_1 = 0$ be any two algebraic curves which are such that $S + \lambda S_1 = 0$ is a curve with one degree of freedom. Let Ω be an operator which changes $S = 0$ into $\Omega S = 0$. Then $\Omega(S + \lambda S_1) = 0$ always passes through the fixed points given by the equations $\Omega S = 0$ and $\Omega S_1 = 0$. If these curves intersect in μ points and if ν be the degree of $\Omega(S + \lambda S_1) = 0$, this has $\frac{1}{2}\nu(\nu+3) - \mu$ degrees of freedom. Now, if σ be the degree of $S = 0$ and σ_1 that of $S_1 = 0$, we must have $\sigma = \sigma_1$, because λ s of dimension zero in x and zero in y. Further $\sigma \not> 2$, because $S + \lambda S_1 = 0$ must pass through $\frac{1}{2}\sigma(\sigma+3) - 1$ points and these must be both on $S = 0$ and $S_1 = 0$; therefore $\frac{1}{2}\sigma(\sigma+3) - 1 = \sigma^2$ or σ is a root of

$$\sigma^2 - 3\sigma + 2 = 0 \quad \text{.....(A).}$$

Hence $\sigma = 1$ or 2, $\sigma = 1$ gives straight lines, $\sigma = 2$ gives conics.

Take $\sigma = 2$. Now, as $\Omega S + \lambda\Omega S_1 = 0$ has a single degree of freedom, the operator Ω can never elevate the degree of $S + \lambda S_1 = 0$ (just proved): *i.e.*, $\Omega = \Omega_1$ is a differential operator. For $\sigma = 1$ the operator $\Omega = \Omega_2$ will be an integrating operator such that $\Omega S + \lambda\Omega S_1 = 0$ must not exceed the degree limit 2. Therefore $\Omega_1\Omega_2 = 1$ in the sense that Ω_1 is the reciprocal process of Ω_2. If we take $\Omega_1 \equiv \xi\,\partial/\partial x + \eta\,\partial/\partial y + \zeta\partial/\partial z$ where $z = \zeta = 1$ after operation and ξ, η are constants, we get the result enunciated by Prof. Cochez in Question 16230.

Thus a plane is a locus of varieties with freedom 2, and as the freedom diminishes to 1 and 0 we gradually pass from the plane to the curve and then to the point. A corresponding process in arithmetic is the passage from the complex domain to the real and then to zero. And, in virtue of the equation (A), conics and straight lines are the only reciprocally related curves in a plane, in the sense that, given one system with one degree of freedom, we can get the other system of equal freedom by operating on the given system with Ω_1 or Ω_2, as the case may be, where $\Omega_1\Omega_2 = 1$.

N.B.—It may perhaps be added that the theorem in Question 16251 W. F. Beard, M.A.) is also deducible from the above considerations.

16205. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Given that $N = x^x$ has 38907 figures, x being an integer, find its first three and last three figures.

Solution by JAMES BLAIKIE, M.A.

$N = x^x$ (38907 figures);

therefore $\log N = x \log x = 38906 + \text{some mantissa}$;

and, if $y = \log x$, $y + \log y = \log 38906 = 4{\cdot}590017$; whence, by trial, we easily find $y = 3{\cdot}98914 = \log x$ approximately. From this, since

x is an integer, we get $x = 9753$ and $\log x = 3{\cdot}989138$; therefore, using six-figure tables,

$$x \log x = 38906{\cdot}063 = \log 116\ldots \text{ (38907 figures).}$$

Thus we have found the first three figures. To find the last three, we may use contracted multiplication, neglecting coefficients of power of 10 greater than the second. Thus we have

$$\begin{aligned} N = 9753^{9753} &= \ldots 753^{8192+1024+512+16+8+1} \\ &= \ldots 041 \times \ldots 481 \times \ldots 441 \times \ldots 721 \times \ldots 561 \times \ldots 753 = \ldots 473; \end{aligned}$$

therefore $N = 116\ldots473$ (38907 figures).

The PROPOSER solves the Question as follows:—

Let C denote the characteristic of $\log N = x \log x$.

Here C = 38906. Taking trial values of x, it will be found (by actual computing) that, when

$$x = 10000,\ 9000,\ 9500,\ 9750,\ 9753,$$

then $C = 40000, 35588, 37788, 38892, 38906.$

Hence $x = 9753$ is the correct value; and, using 7-figure tables, $\log N = x \log x = 38906{\cdot}0648, \ldots$, whence the first three figures of N are 116....

(To find more figures, larger tables must be used.)

Lastly, to find the last three figures of N it suffices to find the residue of x^x to mod 1000. Barlow's *New Mathematical Tables*, 1814, giving 4th and 5th powers of all numbers $\not> 1000$, suffice (after correction of errata) for this purpose. Here $x \equiv 753 \pmod{1000}$. The following table shows the residues (R) of x^n to argument n:—

$n =$	1,	4,	5,	20,	100;	9,	13,	52,	53.
$x^n \equiv$	753,	081,	993,	401,	001;	433,	073,	241,	473.

Hence
$$\begin{aligned} x^x &\equiv 753^{9753} \equiv 753^{(97 \cdot 100 + 53)} = (753^{100})^{97} \cdot 753^{53} \\ &\equiv 753^{53} \equiv 473 \bmod 1000. \end{aligned}$$

Hence $N = 116\ldots473$ (total 38907 figures).

11110. (R. W. D. CHRISTIE.)—Find, by means of the quintic roots of minus unity, all the values in succession of X and Y in the equation $X^2 - 5Y^2 = \pm 4$, and give the values of X and Y respectively.

Solutions (I.) *by the* PROPOSER; (II.) *by* A. H. BELL.

(I.) In $X^2 - 5Y^2 = +4$, let

$$\begin{aligned} X &= (\omega_2 + \omega_3)^{2n} + (\omega_4 + \omega_5)^{2n} = 3, 7, 18, 47, 123, 322, 843, \ldots \\ &\qquad = \text{form } 3a^2 + 2ab + 2b^2, \\ Y &= \frac{(\omega_2 + \omega_3)^{2n} - (\omega_4 + \omega_5)^{2n}}{(\omega_2 + \omega_3) - (\omega_4 + \omega_5)} = 1, 3, 8, 21, 55, 144, \ldots = \text{form } a(a + 2b). \end{aligned}$$

In $5Y^2-X^2=+4$, let

$$X=(\omega_2+\omega_3)^n+(\omega_4+\omega_5)^n = 1, 3, 4, 7, 11, 18, 29, 47, \ldots$$
$$= \text{form } a^2-b^2+4ab,$$

$$Y=\frac{(\omega_2+\omega_3)^n-(\omega_4+\omega_5)^n}{(\omega_2+\omega_3)-(\omega_4+\omega_5)} = 1, 1, 2, 3, 5, 8, \ldots \text{ the continuant series}$$
$$= \text{form } a^2+b^2$$

where $a^2-b^2-ab=\pm 1$ and -1, ω_2, ω_3, ω_4, ω_5 are the roots of $x^5+1=0$.

(II.) When $2x$ and $2y$ are substituted for X and Y this equation becomes $x^2-5y^2=\pm 1$. The quotients for the $\sqrt{5}$ are $2:4$, $4:4$, ... ad inf.; therefore

$$\frac{x_n}{y_n}=\frac{1}{0}, \frac{2}{1}, \ldots \text{ or } \frac{X_n}{Y_n}=\frac{2}{0}, \frac{4}{2}, \ldots.$$

Having two terms, the series is extended by $X_{n+1}=4X_n+X_{n-1}$, and the same for Y. The constant 4 is derived from the quotient 4. Hence the values of

$$\frac{X_n}{Y_n}=\frac{2}{0}, \frac{4}{2}, \frac{18}{8}, \frac{76}{34}, \frac{322}{144}, \frac{1364}{610}, \frac{5778}{2584}, \ldots \text{ ad inf.}$$

16185. (Professor NEUBERG.)—Trouver

$$\int \frac{\sin 5x}{\sin 3x}\,dx, \quad \int \frac{\sin 3x}{\sin 5x}\,dx.$$

Solution by S. NARAYANA AIYAR.

These are only particular cases of the more general integral

$$\int \frac{\sin mx}{\sin nx}\,dx.$$

$$\frac{\sin mx}{\sin nx}=\frac{1}{2n}\Sigma(-1)^k \sin ma \operatorname{cosec}(x-a) \text{ or } \frac{1}{2n}\Sigma(-1)^k \sin ma \cot(x-a),$$

according as $m+n$ is odd or even, where $a=k\pi/n$. Whence the integral is

$$\frac{1}{2n}\Sigma(-1)^k \sin ma \log \tan \tfrac{1}{2}(x-a) \text{ or } \frac{1}{2n}\Sigma(-1)^k \sin ma \log \sin(x-a),$$

according as $m+n$ is odd or even.

Note.—The PROPOSER writes:—"J'avais proposé la question pour provoquer une solution directe qui aurait été préférable. M. Aiyar devrait au moins donner les valeurs explicites des deux intégrales proposées."

16232. (D. BIDDLE.)—A square holding has been staked out on a level part of the prairie. But a fire has invaded the scene, and all traces of the square have vanished, except the charred remains of four of the stakes, known to have been on distinct sides of it. At present they form the vertices of an irregular quadrilateral. One of the owners,

being a mathematician, says that the data are sufficient to enable him to stake out the square afresh on its original lines. How so?

Solutions (I.) *by* W. H. BLYTHE, M.A.; (II.) *by* Professor E. B. ESCOTT; (III.) *by* JULIAN JULIAN; (IV.) *by* HENRY RIDDELL, M.E.

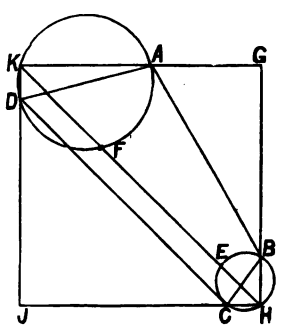
Note.—Solutions (I.) and (II.) are both given on account of the variety of the references.—ED.

(I.) A complete solution of all cases, six in number, will be found in *Solutions to Exercises in Taylor's Euclid*, by W. W. Taylor, No. 326. In this particular case the quadrilateral will not have re-entrant angles, and its angles must not lie on the sides of the square produced.

Let ABCD be the given quadrilateral. Describe circles upon AB and CD. We have then two semicircles external to ABCD and two partly or altogether within the figure. Bisect the semicircles within the figure at the points L and M. Let LM produced cut the external semicircles in S and T; then ST is a diagonal of the required square.

There is but one solution unless ABCD is a square: in this case L and M coincide and we have an infinite number of solutions.

(II.) Under certain conditions the problem may have an indefinite number of solutions. This is a particular case of a more general problem solved in E. Catalan's *Théorèmes et Problèmes de Géométrie élémentaire*, p. 185, Problem 4. This solution is due to Prof. Neuberg.



On AD and BC, the opposite sides of the quadrilateral connecting the four points, as diameters, draw circles. Let F be the middle point of arc AD and E the middle point of arc BC. EF will pass through opposite vertices of the square. The construction is indeterminate when E and F coincide, *i.e.*, when there is a point from which two opposite sides of the quadrilateral subtend a right angle.

(III.) $AB \sin\phi = CD \sin\theta$
$= \text{side of square},$
$\theta - \phi = 90° - \psi.$

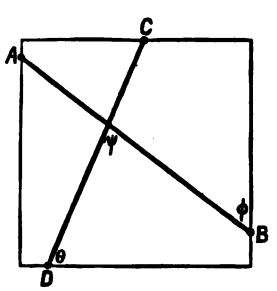
Hence

$AB/CD = \sin\theta/\sin\phi$
$= \cos(\psi - \phi)/\sin\phi$
$= (\cos\psi\cos\phi + \sin\psi\sin\phi)/\sin\phi$
$= \cos\psi\cot\phi + \sin\psi;$

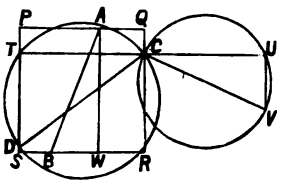
therefore

$(AB/CD - \sin\psi)/\cos\psi = \cot\phi,$

and from this the square may be set out on the original lines. If, however, AB = CD and ψ becomes a right angle, the problem is indeterminate.



(IV.) Let PQRS be a square upon the sides of which A, B, C, D are points chosen at random, one on each side. Join AB, CD (points on pairs of opposite sides). Draw AW parallel to QR, and TC parallel to PQ. Produce TC to U, making CU = TC, and draw UV perpendicular to CU, and equal to BW. Then is CV = AB and perpendicular to it.



Hence the construction for the problem: To describe a square through four points A, B, C, D. At C draw CV = AB and perpendicular to it. Upon CD, CV as diameters describe circles, and through C draw the double chord TC, CU so that TC = CU (a common book exercise). Then is TC = a side of the square and parallel to it, and the method of completing the figure is obvious.

9615. (Professor BORDAGE.)—Find (1) the most general functions f and ϕ such that for all values of x and y the expression

$$V = (x^2-y^2)\,[f(x+y)+\phi(x-y)]$$

can be put under the form $V = F(x)-\Phi(y)$; and (2) the functions F and Φ.

Solution by S. NARAYANAN, B.A., L.T.

We have the differential equations

$$\partial^2 V/(\partial x\,\partial y) = (x^2-y^2)(f''-\phi'')+2x(f'-\phi')-2y(f'+\phi') \text{ from (I.)}$$
$$= 0 \text{ from (II.)},$$

where (I.) and (II.) denote the first and second forms of V.

Hence we should have

$$(x^2-y^2)(f''-\phi'')+2x(f'-\phi')-2y(f'+\phi') = 0.$$

Putting $x+y=k$, $x-y=l$, this becomes

$$d^2f/dk^2+2(df/dk)/k = d^2\phi/dl^2+2(d\phi/dl)/l.$$

Hence $$d^2f/dk^2+2(df/dk)/k = \lambda \text{ (a constant)};$$

therefore $f = a_1+b_1/(x+y)+c(x+y)^2$ and $\phi = a_2+b_2/(x-y)+c(x-y)^2$.

And F and Φ are easily deduced.

16224. (Professor SANJÁNA, M.A.)—Prove that

$$1\,\frac{1}{2}+\frac{1}{2}\,\frac{1.3}{2.4}+\frac{1}{3}\,\frac{1.3.5}{2.4.6}+\frac{1}{4}\,\frac{1.3.5.7}{2.4.6.8}+\ldots = \log_e 4,$$

$$\frac{1}{2}\,\frac{1}{2}+\frac{1}{3}\,\frac{1.3}{2.4}+\frac{1}{4}\,\frac{1.3.5}{2.4.6}+\ldots = 1,$$

$$\frac{1}{3}\,\frac{1}{2}+\frac{1}{4}\,\frac{1.3}{2.4}+\frac{1}{5}\,\frac{1.3.5}{2.4.6}+\ldots = \frac{5}{6}$$

and show how to find the sums of similar series.

Solutions (I.) *by* Professor E. B. Escott; (II.) *by* S. Narayana Aiyar, Mahendra Nath, D.E., M.A., B.Sc., *and* C. M. Ross.

(I.) The series may be summed by the following method, due to Euler :—

Let $$y = \frac{1}{2}x + \frac{1}{2}\cdot\frac{1.3}{2.4}x^2 + \frac{1}{3}\cdot\frac{1.3.5}{2.4.6}x^3 + \ldots,$$

$$\frac{dy}{dx} = \frac{1}{2} + \frac{1.3}{2.4}x + \frac{1.3.5}{2.4.6}x^2 + \ldots,$$

$$x\frac{dy}{dx} = (1-x)^{-\frac{1}{2}} - 1 \quad \text{or} \quad dy = \left(\frac{1}{x(1-x)^{\frac{1}{2}}} - \frac{1}{x}\right)dx,$$

$$y = \log\frac{1}{[1+\sqrt{(1-x)}]^2} + C,$$

and, since $y = 0$ when $x = 0$, $C = \log 4$, $y = \log\frac{4}{[1+\sqrt{(1-x)}]^2}$. When $x = 1$, $y = \log 4$.

The others may be summed in a similar way, or may be summed by the following method given in Chrystal's *Algebra*, Part II., chap. xxvii., § 5 :—

$$\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1.3}{2.4} + \frac{1}{4}\cdot\frac{1.3.5}{2.4.6} + \ldots$$

$$= \frac{\frac{1}{2}}{1.2} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2.3} + \ldots = -\frac{-\frac{1}{2}}{1.2} + \frac{-\frac{1}{2}\cdot\frac{-3}{2}}{1.2.3} - \ldots$$

$$= 2\left(-\frac{\frac{1}{2}\cdot\frac{-1}{2}}{1.2} + \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}}{1.2.3} - \ldots\right) = 2\left[-(1-1)^{\frac{1}{2}} + 1 - \tfrac{1}{2}\right) = 1.$$

$\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1.3}{2.4} + \frac{1}{5}\cdot\frac{1.3.5}{2.4.6} + \ldots$.—The general term

$$\frac{1}{n+2}\cdot\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$$

may be written

$$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots\frac{2n-1}{2}}{1.2.3\ldots n(n+2)} = \frac{-\frac{1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\ldots\cdot\frac{-2n-1}{2}(n+1}{1.2.3\ldots n(n+1)(n+2)}(-1)^n$$

$$= \left(\frac{-\frac{1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\ldots\frac{-2n-1}{2}}{1.2.3\ldots n(n+1)} - \frac{-\frac{1}{2}\cdot\frac{-3}{2}\ldots\frac{-2n-1}{2}}{1.2.3\ldots(n+2)}\right)(-1)^n;$$

therefore the sum of the series is

$$\frac{4}{3}\left(-(1-1)^{\frac{1}{2}} + 1 - \frac{3}{2} + \frac{3.1}{2.4}\right) + 2\left(-(1-1)^{\frac{1}{2}} + 1 - \frac{1}{2}\right) = \frac{5}{6}.$$

(II.) By the binomial theorem

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots.$$

Multiplying both sides by x^n where n is positive and integrating both sides within the limits 1 and 0, we get

$$\int_0^1 x^n (1-x)^{-\frac{1}{2}} dx = \left[\frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2}\cdot\frac{1}{2} + \frac{x^{n+3}}{n+3}\cdot\frac{1.3}{2.4} + \ldots\right]_0^1$$

$$= \frac{1}{n+1} + \frac{1}{n+2}\cdot\frac{1}{2} + \frac{1}{n+3}\cdot\frac{1.3}{2.4} + \frac{1}{n+4}\cdot\frac{1.3.5}{2.4.6} + \ldots .$$

Denoting the series $\frac{1}{n+2}\cdot\frac{1}{2} + \frac{1}{n+3}\cdot\frac{1.3}{2.4} + \frac{1}{n+4}\cdot\frac{1.3.5}{2.4.6} + \ldots$ by S_n, we get

$$S_n = \int_0 x^n (1-x)^{-\frac{1}{2}} dx - \frac{1}{n+1} = \frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{\Gamma\left(\frac{2n+3}{2}\right)} - \frac{1}{n+1}.$$

By giving to n successively the values 0, 1, 2, ..., we get the values of the series beginning with the second. To get the first series multiply by $1/x$ and then integrate.

Note on the Empirical Rule of Mr. Saradakanta Ganguli, M.A.

By Professor Sanjána, M.A.

A better approximation is given by the following method, which also explains the rule. By the ordinary formula, $M = P(1+r/100)^n$; here $M = 3P$; so that $3 = (1+r/100)^n$. Therefore we get

$$n\left(\frac{r}{100} - \frac{1}{2}\frac{r^2}{100^2} + \frac{1}{3}\frac{r^3}{100^3}\ldots\right) = 1{\cdot}0986;$$

i.e., $$nr\left(1 - \frac{1}{2}\frac{r}{100}\ldots\right) = 109{\cdot}86 = 110 \text{ (very nearly).}$$

Thus $nr = 110\left(1 - \frac{1}{2}\frac{r}{100}\right)^{-1}$ (approximately) $= 110\left(1 + \frac{1}{2}\frac{r}{100}\right)$;

so that $$n = \frac{110}{r} + {\cdot}55.$$

When $r = 1, 2, 3, 4$, $n = \frac{110{\cdot}55}{1}, \frac{111{\cdot}1}{2}, \frac{111{\cdot}65}{3}, \frac{112{\cdot}2}{4}$ respectively.

Again $$r = \frac{110}{n - {\cdot}55} = \frac{110}{n} + \frac{60}{n^2} \text{ (roughly);}$$

whence, n being given, r may be calculated. Similar rules may be constructed for doubling, quadrupling, &c., the principal, using natural logarithms.

16208. (Professor Nanson.)—If $a^n + b^n + c^n = 0$, $a^{n-1}x + b^{n-1}y + c^{n-1}z = 0$, $x^n + y^n + z^n = 0$, and $abc \neq 0$, then either x/a, y/b, z/c are all equal or the sum of their homogeneous products of $n-2$ dimensions is zero.

Solution by S. NARAYANAN, W. F. BEARD, M.A., *and* M. T. NARANIENGAR, M.A.

The equations when written in the form

$$\left.\begin{aligned} a^n.1 + b^n.1 + c^n.1 &= 0 \\ a^n.x/a + b^n.y/b + c^n.z/c &= 0 \\ a^n.(x/a)^n + b^n.(y/b)^n + c^n.(z/c)^n &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

may be considered simultaneous in a^n, b^n, c^n.

Hence, either $a = b = c = 0$ (which is disallowed by the hypothesis), or the determinant of (A),

$$\begin{vmatrix} 1, & 1, & 1 \\ \lambda, & \mu, & \nu \\ \lambda^n, & \mu^n, & \nu^n \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

where $\lambda = x/a, \quad \mu = y/b, \quad \nu = z/c.$

The factors of (B) are $(\lambda-\mu)(\mu-\nu)(\nu-\lambda)\,H_{n-2}$, where H_{n-2} denotes a homogeneous function in λ, μ, ν of $n-2$ dimensions.

If $\lambda = \mu$, we have $x/a = y/b \equiv k$; therefore $x = ka$, $y = kb$. Substituting these in the third of the given equations, we get

$$k^n a^n + k^n b^n + z^n = 0.$$

And the first given equation gives

$$k^n a^n + k^n b^n + k^n c^n = 0.$$

Hence $z^n = k^n c^n$; therefore $z/c = k$; therefore $\lambda = \mu = \nu$, or $H_{n-2} = 0$.

16230. (Professor COCHEZ.)—On considère toutes les coniques passant par quatre points donnés. Démontrer que les polaires d'un point fixe par rapport à toutes ces coniques passent au même point.

Solutions (I.) *by* HENRY RIDDELL, M.E.; (II.) *by* L. B. BENNY *and* Professor SANJÁNA, M.A.

(I.) This is a theorem proved in many of the text-books. The following proof is about as short as can be given :—

By projecting two of the points of intersection into the two circular points, we have the system of conics transformed into a system of co-axial circles. The polars of a fixed point for this system of circles obviously pass through the point diametrically opposite the given point on the circle of the conjugate co-axial system which passes through the fixed point d. For this circle cuts all the system at right angles, and the line joining its centre to that of any circle of the system is cut in harmonic ratio by the two circumferences.

(II.) Take the diagonal triangle of a quadrangle formed by four points for triangle of reference. The four points are expressible as $\pm f, \pm g, \pm h$). The equation of any conic through the four points is

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0 \quad \text{with condition} \quad uf^2 + vg^2 + wh^2 = 0 \ldots\ldots(1).$$

The polar of the fixed point $(\alpha_0, \beta_0, \gamma_0)$ with reference to any of these conics has the equation $u\alpha\alpha_0 + v\beta\beta_0 + w\gamma\gamma_0 = 0$,
and condition (1) shows that this always passes through the fixed point $(f^2/\alpha_0, g^2/\beta_0, h^2/\gamma_0)$.

16041. (Professor SANJÁNA, M.A., and M. V. ARUNACHALAN, M.A.)—If $R_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n$, prove that $R_1 - R_2 + R_3 - R_4 + \ldots$ ad inf. $= \frac{1}{2}\log 2$.

Note by G. H. HARDY, M.A.

The series $R_1 - R_2 + R_3 - \ldots$ where

$$R_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + 1/n$$

is obviously not convergent. If, however, S_n denotes the sum of the first n terms, $s_{2n} = -\frac{1}{2} - \frac{1}{4} - \ldots - 1/2n$, $s_{2n-1} = 1 + \frac{1}{3} + \ldots + 1/(2n+1)$,

and $$(s_1 + s_2 + \ldots + s_{2n})/2n = (\sigma_2 + \sigma_4 + \ldots + \sigma_{2n})/2n,$$

where $$\sigma_{2n} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots + 1/(2n-1) - 1/2n,$$

Since $\lim \sigma_{2n} = \log 2$, it follows, by a well known theorem, that

$$\lim (s_1 + s_2 + \ldots + s_{2n})/2n = \tfrac{1}{2}\log 2;$$

and, since $\lim s_n/n = 0$, we can say simply that

$$\lim (s_1 + s_2 + \ldots + s_n)/n = \tfrac{1}{2}\log 2.$$

Thus the series, though not convergent, is summable, in the sense defined by Frobenius and Césaro, and its "sum" is $\frac{1}{2}\log 2$.

16190. (R. W. D. CHRISTIE.)—If N = AB, all integral, prove that the small factor $A = \{[4b\mathrm{N} + (a-bc)^2]^{\frac{1}{2}} - a + bc\}/2b$ and the large factor $B = \frac{1}{2}\{[4b\mathrm{N} + (a-bc)^2]^{\frac{1}{2}} + a - bc\}$. Also $(a-bc)^2 = (\mathrm{B} - b\mathrm{A})^2$, all the letters being integral: *e.g.*, N = 257821. Then one set of values (out of many) is $b = 2$, $c = 10$, $a = 69$; or, again, $b = 2$, $c = 8$, $a = 65$. It is required to find the small letters.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let $$s = \{4b\mathrm{N} + (a-bc)^2\}^{\frac{1}{2}} \text{ and } a - bc = d \quad \ldots\ldots (1).$$

Then $$\mathrm{A} = (s-d)/2b, \quad \mathrm{B} = \tfrac{1}{2}(s+d) \quad \ldots\ldots (2),$$

and these give AB = N identically. Also, from (2), A < B (provided $b > 1$); so that A will be the smaller factor if $b > 1$.

Equation (2) gives at once $\mathrm{B} - b\mathrm{A} = d = a - bc$ by eliminating s; but s as defined in (1) involves a double sign; the ambiguity is removed by squaring: thus giving

$$(\mathrm{B} - b\mathrm{A})^2 = (a-bc)^2 \quad \ldots\ldots (3).$$

Lastly, a, b, c may be any numbers satisfying the condition (3) in such a way that A, B given by (2) are both integers, and also $b > 1$.

But a, b, c cannot be generally determined in any easy way unless A, B be given (as this would involve the whole question of factorization). If A, B be given, then, by (1), (2), (3),

$$(bA + B)^2 = z^2 = [4bN + (B - bA^2)] \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

a 2-ic equation for finding b.

Substituting either of these values into (3) gives a linear relation between a, c. Any values of a, c satisfying this will suffice. Thus, when A, B are given, b is definite; a, c are many-valued.

16206. (T. STUART, M.A., D.Sc.)—The number 10091401 has been proved prime by both Euler and Legendre, but only after tedious calculations. Obtain the same result by calculations not requiring more than an hour's work.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

This number (if prime) can be expressed in each of the 2-ic forms $N = t^2 + D.u^2$, where

D = 1, 2, 3, 5, 6, 10, 15, 22, 30, 33, 57, 60, 72, 165, 190, 330, 760, 1320.

All these numbers (D) are positive idoneals, *i.e.*, are such that, if N is expressible in only one way in the above form, this proves N prime (or a power of a prime).

Now $N = 1251^2 + 2920^2$ is unique; hence N is prime: this was Euler's proof.

A much easier partition, however, is $N = t^2 + 1320u^2$, because here u cannot be greater than 87. And the number of trials can be greatly reduced by use of a few test-moduli, say, $q = 7, 13, 17$.

For the only residues possible for squares (such as t^2, u^2) with these moduli are

$$t^2 \text{ and } u^2 \equiv 0, 1, 2, 4 \pmod 7 \equiv 0, 1, 3, 4, 9, 10, 12 \pmod{13}$$
$$\equiv 0, 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17}.$$

Also $N \equiv 5 \pmod 7 \equiv 8 \pmod{13} \equiv 14 \pmod{17}$,
$D = 1320 \equiv 4 \pmod 7 \equiv 7 \pmod{13} \equiv 11 \pmod{17}$.

Hence $t^2 + 4u^2 \equiv 5 \pmod 7$, $t^2 + 7u^2 = 8 \pmod{13}$,

and $t^2 + 11u^2 \equiv 14 \pmod{17}$.

The only values of t^2, u^2 satisfying these congruences give for u^2

$$u^2 = 1, 2 \pmod 7 \equiv 1, 3, 9, 10 \pmod{13} \equiv 2, 4, 9, 15, 16 \pmod{17},$$

whence $u = 7i \pm 1, \pm 3 = 13i \pm 1, 3, 4, 6 = 17i \pm 2, 3, 4, 6, 7$.

Hence the only possible values of u are

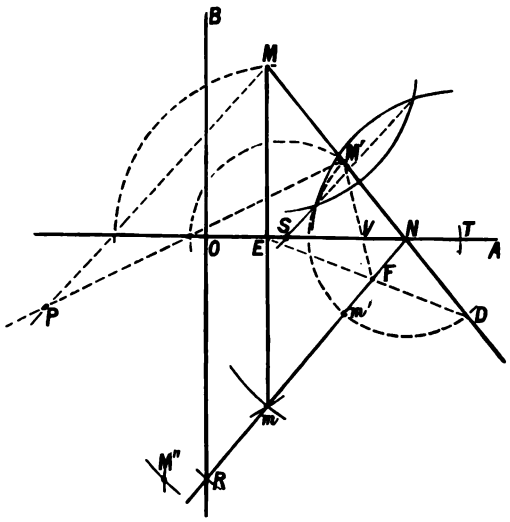
$$u = 3, 4, 6, 10, 20, 27, 32, 36, 38, 48, 53, 55, 64, 66, 71, 81, 87;$$

and, on actual trial of the values of $(N - 1320u^2)$ with each of them, only one square results, viz., $(N - 1320.38^2) = 2861^2$; so that $N = 2861^2 + 1320.38^2$ is unique. This proves N prime (not being a power of a prime).

16071. (Professor NEUBERG.)—D'un point donné P on peut mener à une parabole Σ trois normales PM, PM', PM''; les points d'incidence M, M', M'' sont situés sur une hyperbole équilatère Σ' passant par P. Etant donnés cette hyperbole et les points M, M', construire le point M'', le point P et le sommet de la parabole.

Solutions (I.) *by* C. E. YOUNGMAN, M.A.; (II.) *by* C. M. ROSS.

(I.) Let OA, OB be the asymptotes of the hyperbola; one of them, say OA, is also the axis of the parabola. Draw MM', cutting OA at N, and on it take ND = NM'; find m, the image of M in OA, and draw



Mm, cutting OA at E; find F, where Nm meets DE. Then FM' cuts OA at V, the vertex of the parabola; for DEF is the Pascal line of the hexagon M'M$mm'\infty$V, m' being the image of M' in OA, and ∞ the infinite vertex of the parabola.

Let Nm meet OB at R; mR = Nm'; find M'', the fourth corner of the parallelogram VNRM''. Then MM' and M''V are equally inclined to OA, and the distance of M'' from OA is twice that of the mid-point of mm' or MM'; so that the centroid of MM'M'' lies on OA; and these two properties identify M'' with the required third point of the co-normal set MM'M''.

On OA take VT = VE, and find S equidistant from T and M. Then S is the focus of the parabola, and makes it easy to draw the normals at M and M', giving P.

The operation formula for the whole construction is

$$16R_1 + 8R_2 + 16C_1 + C_2 + 10C_3 = 51.$$

(II.) Let the parabola Σ be $y^2 = 4ax$ ..... (1)
and P (h, k); then any normal to the curve is

$$y - y' = -(y'/2a)(x - x') \quad \text{..... (2).}$$

This passes through (h, k); therefore

$$k-y' = -(y'/2a)(h-x') \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Hence the locus of (x', y') is the rectangular hyperbola Σ', of which one asymptote is the x-axis and whose centre has for its abscissa $(h-2a)$;

or $$\Sigma' \equiv xy = (h-2a)y + 2ak \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Now the abscissæ of the feet of the three normals are given by the x-eliminant of Σ and Σ'.

From (4), $$y = 2ak/(x-h+2a);$$

therefore (1) is $$ak^2 = x(x-h+2a)^2$$

or $$x^3-2x^2(h-2a)+x(h-2a)^2-ak^2 = 0.$$

If x_1, x_2, x_3 are the roots of this equation,

$$\Sigma(x_1) = 2(h-2a), \quad \Sigma(x_1x_2) = (h-2a)^2, \quad x_1x_2x_3 = ak^2 \quad \ldots\ (\alpha).$$

Similarly, the ordinates of the feet are given by

$$y^3-4ay(h-2a)-8a^2k = 0.$$

If y_1, y_2, y_3 are the roots,

$$\Sigma(y_1) = 0, \quad \Sigma(y_1y_2) = -4a(h-2a), \quad y_1y_2y_3 = 8a^2k \quad \ldots\ldots\ (\beta).$$

From (α), $(\Sigma x_1)^2 = 4\Sigma x_1x_2$; therefore $\Sigma x_1^2 - 2\Sigma x_1x_2 = 0$.

Now the hyperbola Σ' is given; so are M and M'. Therefore

$$x_3^2-2x_3(x_1+x_2)+(x_1-x_2)^2 = 0$$

gives the abscissa of M'' or $x_3 = (\sqrt{x_1} \pm \sqrt{x_2})^2$; and, from (β),

$$y_3 = -(y_1+y_2).$$

From (α) and (β) we get a and k.

$$(x_1x_2x_3)^{\frac{1}{2}}/(y_1y_2y_3)^{\frac{1}{2}} = \tfrac{1}{2}k \text{ and } (y_1y_2y_3)^{\frac{1}{2}}/(x_1x_2x_3)^{\frac{1}{2}} = 4a.$$

$$2h = x_1+x_2+x_3+4a = x_1+x_2+x_3+(y_1y_2y_3)^{\frac{1}{2}}/(x_1x_2x_3)^{\frac{1}{2}}.$$

The values of x_3 and y_3 can be easily substituted.

Thus the points are found.

16220. (Hon. George R. Dick, M.A.)—Show that the determinant

$$\begin{vmatrix} \omega_1+\omega_2-\omega_4-\omega_5, & \omega_1\omega_2-\omega_4\omega_5, & \omega_1\omega_2(\omega_4+\omega_5)-\omega_4\omega_5(\omega_1+\omega_2) \\ \omega_2+\omega_3-\omega_5-\omega_6, & \omega_2\omega_3-\omega_5\omega_6, & \omega_2\omega_3(\omega_5+\omega_6)-\omega_5\omega_6(\omega_2+\omega_3) \\ \omega_3+\omega_4-\omega_6-\omega_1, & \omega_3\omega_4-\omega_6\omega_1, & \omega_3\omega_4(\omega_6+\omega_1)-\omega_6\omega_1(\omega_3+\omega_4) \end{vmatrix}$$

vanishes identically. By attributing an appropriate geometrical significance to the ω's, deduce the truth of Brianchon's theorem.

Solution by S. T. Shovelton, M.A., *and* C. M. Ross.

Multiply the rows of the determinant by $\omega_3-\omega_6$, $\omega_4-\omega_1$, $\omega_5-\omega_2$ respectively, and add the columns. Each sum is zero, and therefore the determinant vanishes identically.

Take six tangents to the parabola $y^2 = 4ax$, $y = x/\omega_1 + a\omega_1, \ldots$. The first and second meet at $[a\omega_1\omega_2, a(\omega_1+\omega_2)]$. The line joining the meet of the first and second to that of the fourth and fifth is

$$x[\omega_4+\omega_5-\omega_1-\omega_2]-y[\omega_4\omega_5-\omega_1\omega_2]+a[\omega_1\omega_2(\omega_4+\omega_5)-\omega_4\omega_5(\omega_1+\omega_2)] = 0.$$

Similarly for the other joining lines. The determinant shows that these lines are concurrent. Hence, by projection, Brianchon's theorem is proved for any conic.

Note.—The PROPOSER remarks :—

The geometrical property may be obtained without having recourse to projection by taking the equation of the conic in the form $y^2-2x=0$, so that $x=2\omega_1^2$, $y=2\omega_1$ for the point ω_1, &c.

15541. (Rev. T. ROACH, M.A.)—Sum the series

(1) $\frac{1}{1.2}-\frac{1}{5.6}+\frac{1}{9.10}-\ldots$, (2) $\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\ldots$.

Note on the Latter Series $\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\ldots$ *by* Dr. EMIL LAMPE.

1. The general term of this series is $n/(3n-1)$, which tends to $\frac{1}{3}$ for $n=\infty$. Therefore the series is non-convergent, but oscillates between two values S_1 and S_2. Putting

$$S_1=\left(\frac{1}{2}-\frac{2}{5}\right)+\left(\frac{3}{8}-\frac{4}{11}\right)+\left(\frac{5}{14}-\frac{6}{17}\right)+\ldots=\frac{1}{2.5}+\frac{1}{8.11}+\frac{1}{14.17}+\ldots,$$

$$S_2=\frac{1}{2}-\left(\frac{2}{5}-\frac{3}{8}\right)-\left(\frac{4}{11}-\frac{5}{14}\right)-\left(\frac{6}{17}-\frac{7}{20}\right)-\ldots$$

$$=\frac{1}{2}-\left(\frac{1}{5.8}+\frac{1}{11.14}+\frac{1}{17.20}+\ldots\right).$$

we have $S_2=S_1+\frac{1}{3}$.

2. *Determination of* S_1.—Evidently

$$\int_0^1\frac{x\,dx}{1+x^3}=\tfrac{1}{2}-\tfrac{1}{5}+\tfrac{1}{8}-\tfrac{1}{11}+\ldots=3S_1,$$

$$\int\frac{x\,dx}{1+x^3}=\tfrac{1}{3}\left(\log\frac{\sqrt{(x^2-x+1)}}{x+1}+\sqrt{3}\text{ arc tan}\frac{2x-1}{\sqrt{3}}\right),$$

whence $3S_1=\frac{1}{3}(-\log 2+\frac{1}{3}\pi\sqrt{3})$, $S_1=\frac{1}{27}(-\log 8+\pi\sqrt{3})$.

3. *Direct Determination of* $\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\ldots$.—Having

$$x/(1+x^3)^2=x-2x^4+3x^7-4x^{10}+\ldots,$$

we find

$$\int_0^1\frac{x\,dx}{(1+x^3)^2}=\tfrac{1}{2}-\tfrac{2}{5}+\tfrac{3}{8}-\tfrac{4}{11}+\ldots,$$

provided we admit of this substitution of the upper and lower limit in the series, though it does not remain convergent by this operation.

Decomposing the fraction, we get

$$\frac{x}{(1+x^3)^2}=\tfrac{1}{9}\left(-\frac{1}{x+1}-\frac{1}{(x+1)^2}+\frac{x-1}{x^2-x+1}+\frac{3}{(x^2-x+1)^2}\right),$$

and, integrating,

$$\int_0^1\frac{x\,dx}{(1+x^3)^2}=\tfrac{1}{9}\left(\log\frac{\sqrt{(x^2-x+1)}}{x+1}+\frac{3x^2}{1+x^3}+\sqrt{3}\text{ arc tan}\frac{2x-1}{\sqrt{3}}\right)_0^1$$

$$=\tfrac{1}{9}(\tfrac{3}{2}-\log 2+\tfrac{1}{3}\pi\sqrt{3})=\tfrac{1}{6}+\tfrac{1}{27}(\pi\sqrt{3}-\log 8).$$

Thus we obtain for the value required

$$\tfrac{1}{2}-\tfrac{2}{5}+\tfrac{3}{8}-\tfrac{4}{11}+\ldots=\tfrac{1}{6}+\tfrac{1}{27}(\pi\sqrt{3}-\log 8),$$

and this value proves to be $=\frac{1}{2}(S_1+S_2)$.

4. *Explanation of the Divergence in these Results.*—Considering the series $\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\ldots$ as the limiting value of

$$\tfrac{1}{2}x^2-\tfrac{2}{5}x^5+\tfrac{3}{8}x^8-\tfrac{4}{11}x^{11}+\ldots$$

for $x=1$, we must remember that this power series converges for any positive value of x less than unity, say for $x=1-\epsilon$, ϵ being an arbitrarily small positive quantity. Taking for granted that for $\epsilon=0$ the limiting value is reached, then the sum is $\frac{1}{2}(S_1+S_2)$. Compare a theorem of Frobenius in *Journal für die reine und angewandte Mathematik*, Vol. LXXXIX.

The case is just the same as with the Leibnizian series

$$1/(1+x)=1-x+x^2-x^3+\ldots$$

for $x=1$. Considered as the limit of this power series, the sum $1-1+1-1+\ldots$ has the value of $\frac{1}{2}$. Compare *Encyklopädie der math. Wissensch.*, Vol. I., p. 106–108.

After these models it is easy to form similar questions.

16007. (Professor NANSON.) —Eliminate x, y from

$$x^2=ax+by+c,\quad xy=a'x+b'y+c',\quad y^2=a''x+b''y+c''.$$

Solution by Hon. G. R. DICK, M.A., *and* A. M. NESBITT, M.A.

This Question appears, at first sight, to be a particular case of the general problem of eliminating x and y from any three equations of the second degree in x and y. Taking the general form, let

$$\left.\begin{aligned}a_1x^2+b_1xy+c_1y^2+d_1x+e_1y+f_1&=0\\ a_2x^2+b_2xy+c_2y^2+d_2x+e_2y+f_2&=0\\ a_3x^2+b_3xy+c_3y^2+d_3x+e_3y+f_3&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

be three such equations. Solving for x^2, xy, y^2 in terms of x, y, we get three equations of the type given, *i.e.*,

$$x^2=ax+by+c,\quad xy=a'x+b'y+c',\quad y^2=a''x+b''y+c''\ldots(1,\ 2,\ 3).$$

Any process therefore by which the eliminant of these three equations can be formed may be applied when the system (A) is given.

The usual well known method is given in Salmon's *Higher Algebra*, p. 78, 3rd edition, and may be applied to the case under consideration. But it will be interesting, and, I believe, useful, to find the eliminant by a different procedure.

We have at once

$$(ax+by+c)(a''x+b''y+c'')=(a'x+b'y+c')^2$$

or $x^2(aa''-a'^2)+xy(ab''+a''b-2a'b')+y^2(bb''-b'^2)$
$+x(ac''+a''c-2a'c')+y(bc''+b''c-2b'c')+cc''-c'^2=0.$

Replacing x^2, xy, y^2 by their given expressions in terms of x and y, we get the linear relation $\quad Ax+By+C=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a),$

wherein $A = a^2a'' - aa'^2 + aa'b'' + a'a''b - 2a'^2b' + a''bb'' - a''b'^2$,
$B = aa''b - a'^2b + ab'b'' + a''bb' - 2a'b'^2 + bb''^2 - b'^2b''$,
$C = aa''c - a'^2c + ab''c' + a''bc' - 2a'b'c' + bb''c'' - b'^2c''$.

Now, multiply (α) by x and y successively and replace as before, and we get
$x(Aa + Ba' + C) + y(Ab + Bb') + Ac + Bc' = 0$ .....(β),
$x(Aa' + Ba'') + y(Ab' + Bb'' + C) + Ac' + Bc'' = 0$ .....(γ).

Thus, from the linear equations (α), (β), (γ) we get the eliminant in the form of a determinant

$$\begin{vmatrix} A, & Aa + Ba' + C, & Aa' + Ba'' \\ B, & Ab + Bb', & Ab' + Bb + C \\ C, & Ac + Bc', & Ac' + Bc'' \end{vmatrix} = 0 \text{ .....}(\delta),$$

which may sometimes be convenient. It is noticeable that, if any two of the quantities A, B, C vanish, so does the third.

The eliminant may also be found by expressing y in terms of x from the equation $xy = a'x + b'y + c'$ and substituting in the two others, thus producing two cubics in x. The eliminant may then be found as in Salmon, *Higher Algebra*, p. 72, 3rd edition; but I think the form (δ) is preferable.

16210. (A. M. Nesbitt, M.A.) — Prove from first principles that the locus of a point whence a straight line can be drawn to intersect $(x-a_r)/l_r = (y-\beta_r)/m_r = (z-\gamma_r)/n_r$ $(r = 1, 2, 3)$ is $|P_1, Q_2, R_3| = 0$, where P_k, Q_k, R_k are the co-factors of p_k, q_k, r_k in the determinant

$$\begin{vmatrix} p_k, & q_k, & r_k \\ l_k, & m_k, & n_k \\ x-a_k, & y-\beta_k, & z-\gamma_k \end{vmatrix}.$$

[Note that the locus is a quadric and that the three given lines are obviously generators.]

Solution by Lalit Mohan Mukherji, M.A., B.L.

Let (α, β, γ) be the point and $(x-\alpha)/L = (y-\beta)/M = (z-\gamma)/N$ the equation of any straight line through it which intersects all the given straight lines. Then we have

$$\begin{vmatrix} \alpha-a_1, & \beta-\beta_1, & \gamma-\gamma_1 \\ l_1, & m_1, & n_1 \\ L, & M, & N \end{vmatrix} = 0$$

and two similar determinants equated to zero as the conditions of intersection. Expanding these in terms of L, M, and N, we see that the coefficients of L, M, and N are P_1, Q_1, R_1 in the first, P_2, Q_2, R_2 in the second, and P_3, Q_3, R_3 in the third.

Eliminating L, M, N from these three equations, we get

$$\begin{vmatrix} P_1, & P_2, & P_3 \\ Q_1, & Q_2, & Q_3 \\ R_1, & R_2, & R_3 \end{vmatrix} = 0 \quad \text{or} \quad |P_1Q_2R_3| = 0.$$

Note on Expansion in Series.

By S. NARAYANA AIYAR.

Let $f(x)$ be capable of expansion in powers of x and be equal to $a_0 + \Sigma a_n x^n$. Then

$\frac{1}{2}[f(x+iy)+f(x-iy)]$ or $C(x, y) = a_0 + \Sigma a_n (x^2+y^2)^{\frac{1}{2}n} \cos n \tan^{-1} y/x$, and

$1/2i\,[f(x+iy)-f(x-iy)]$ or $S(x, y) = \Sigma a_n (x^2+y^2)^{\frac{1}{2}n} \sin n \tan^{-1} y/x$.

These can be proved as follows:—substitute successively $x+iy$ and $x-iy$ on both sides for x; add or subtract, as the case may be, and divide by 2 or $2i$. Then the n-th term in dexter is

$\frac{1}{2}a_n[(x+iy)^n + (x-iy)^n]$ or $(1/2i)\,a_n[(x+iy)^n - (x-iy)^n]$,

which, by De Moivre's theorem, reduces to

$a_n (x^2+y^2)^{\frac{1}{2}n} \cos n \tan^{-1} y/x$ or $a_n (x^2+y^2)^{\frac{1}{2}n} \sin n \tan^{-1} y/x$,

as the case may be.

By the application of the above theorem many elegant results in expansions can be established. For example:—

(1) If $f(x) = \sin x = \sum_{n=0}^{n=\infty} \frac{x^n}{n!} \sin \frac{1}{2}n\pi$, then

$$C(x, y) = \sin x \cosh y = \Sigma \frac{(x^2+y^2)^{\frac{1}{2}n}}{n!} \sin \tfrac{1}{2}n\pi \cos n \tan^{-1} y/x,$$

$$S(x, y) = \cos x \sinh y = \Sigma \frac{(x^2+y^2)^{\frac{1}{2}n}}{n!} \sin \tfrac{1}{2}n\pi \sin n \tan^{-1} y/x.$$

(2) If $f(x) = \cos x = \sum_{n=0}^{n=\infty} \frac{x^n}{n!} \cos \frac{1}{2}n\pi$, then

$$\left.\begin{matrix} C(x, y) = \cos x \cosh y \\ S(x, y) = -\sin x \sinh y \end{matrix}\right\} = \sum_{n=0}^{n=\infty} \frac{(x^2+y^2)^{\frac{1}{2}n}}{n!} \cos \tfrac{1}{2}n\pi \begin{matrix}\cos \\ \sin\end{matrix} n \tan^{-1} y/x.$$

(3) If $f(x) = \tan x = \Sigma \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n-1} x^{2n-1}$, then

$$\left.\begin{matrix} C(x, y) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y} \\ S(x, y) = \dfrac{\sinh 2y}{\cos 2x + \cosh 2y} \end{matrix}\right\}$$

$$= \Sigma \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n-1} (x^2+y^2)^{\frac{1}{2}(2n-1)} \begin{matrix}\cos \\ \sin\end{matrix} (2n-1) \tan^{-1} y/x.$$

(4) $f(x) = \cot x = 1/x - \Sigma \frac{2^{2n}}{(2n)!} B_{2n-1} x^{2n-1}$.

$$\left.\begin{matrix} C(x, y) = \dfrac{\sin 2x}{\cosh 2y - \cos 2x} \\ S(x, y) = \dfrac{\sinh 2y}{\cos 2x - \cosh 2y} \end{matrix}\right\}$$

$$= \frac{2}{x^2+y^2} \left\{\begin{matrix} x \\ -y \end{matrix}\right\} + \Sigma \frac{2^{2n}}{(2n)!} B_{2n-1} (x^2+y^2)^{\frac{1}{2}(2n-1)} \begin{matrix}\cos \\ \sin\end{matrix} (2n-1) \tan^{-1} y/x.$$

[In (3) and (4), B_{2n-1} is the n-th Bernoulli's number.]

Similar results hold when $f(x) = \sec x$, $\operatorname{cosec} x$.

The function $f(x)$ may be of any form. If it is e^x, the

$$(5)\qquad \left.\begin{matrix} C(x,y) \\ S(x,y) \end{matrix}\right\} = e^x \begin{matrix}\cos\\ \sin\end{matrix} y = \Sigma \frac{(x^2+y^2)^{\frac{1}{2}n}}{n!} \begin{matrix}\cos\\ \sin\end{matrix} n \tan^{-1} y/x.$$

By putting $x = ax$ and $y = bx$, we get the particular cases

$$e^{ax} \begin{matrix}\cos\\ \sin\end{matrix} bx = \Sigma \frac{(a^2+b^2)^{\frac{1}{2}n}}{n!} x^n \begin{matrix}\cos\\ \sin\end{matrix} n \tan^{-1} b/a.$$

(6) $f(x) = 1/(1-x) = 1 + \Sigma x^n$, when x is < 1.

$$\left.\begin{aligned} C(x,y) &= \frac{1-x}{1-2x+x^2+y^2} \\ S(x,y) &= \frac{y}{1-2x+x^2+y^2} \end{aligned}\right\} = 1 + \Sigma (x^2+y^2)^{\frac{1}{2}n} \begin{matrix}\cos\\ \sin\end{matrix} n \tan^{-1} y/x.$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we get the well known expansions in plane trigonometry

$$\left.\begin{aligned} \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} &= 1 + \Sigma r^n \cos n\theta \\ \frac{r\sin\theta}{1-2r\cos\theta+r^2} &= 1 + \Sigma r^n \sin n\theta \end{aligned}\right\} \text{ where } r \text{ is less than 1.}$$

(7) When $f(x) = x^n$, putting $x = y$ in $C(x, y)$ and $S(x, y)$ by the binomial theorem, we get the values of

$$1 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots, \qquad {}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \ldots.$$

Take $f(x)$ an inverse function. For example, take $f(x) = \tan^{-1} x$; then we get a series corresponding to Gregory's series.

The examples may be multiplied by taking different forms of $f(x)$.

A Demonstration of the Transcendental Character of the Function $e^{H(z)}$.

By F. Tavani.

The function $e^{H(z)}$ has not a finite number of determined roots. For, if $e^{H(z)}$ has a finite number of roots (or zeros), let these be $a_1, a_2, \ldots, a_n$. We shall have then $e^{H(a_m)} = 0$ .....(1).

Now it is evident that all the possible zeros of $e^{H(z)}$ are not zeros of $H(z)$; therefore any possible root of $e^{H(z)}$ is the root of a function of the type $G(z)/H(z)$, and, dividing $e^{H(z)}$ by $G(z)/H(z)$, we clear $e^{H(z)}$ of some of its possible roots; and, if $e^{H(z)}$ has a finite number of roots, by applying this process a finite number of times (*e.g.*, n times), we shall obtain a function free of all of the roots in finite number of $e^{H(z)}$ having the form

$$[H(z)]_n / [G_1(z)\, G_2(z) \ldots G^n(z)]\, e^{H(z)},$$

or, for brevity, $L(z)\, e^{H(z)}$ .....(2),

so that $e^{H(a)} L(a) \neq 0$, $e^{H(a)} = A/[L(a)]$ ..... (3, 4).

The equation (4) shows that a is not a zero of $e^{H(z)}$, unless it is a pole of $L(z)$, and consequently of $H(z)$ also. Now there is an indefinite number of poles of $L(z)$, which are the poles of $H(z)$ and the roots of $\prod_{i=1}^{n} G_i(z)$; these roots are quite arbitrary, since the functions $G_i(z)$ are arbitrary, and may be chosen so as to contain an infinity of arbitrary roots, amongst which we have noticed to be the possible roots $a_1, a_2, \ldots, a_n$ of $e^{H(z)}$. Therefore, $e^{H(z)}$ if it is susceptible of a root a, is susceptible of an infinity of indeterminate roots also. It is well known that the poles of $H(z)$, which belong to the infinity of such roots, constitute essential singularities of $e^{H(z)}$.

If a is not a pole of $L(z)$, then the equation (4) shows that it is not a zero of $e^{H(z)}$ either, as it had been supposed in (1); and, in the same manner, any other quantity is excluded from being a root of $e^{H(z)}$.

In conclusion of these few remarks, it must be shown that it is not only the infinite number of roots—as it is suggested in some text-book on the theory of functions—that constitutes the character of transcendentality; but their infinity accompanied by their indeterminateness, and by virtue of both these characteristics, and not of the first one only, they are analytically representable by a function possessing the factor $e^{H(z)}$, or, what is the same, they possess an essential singularity, which constitutes the differential character of transcendental functions from algebraical ones.

16101. (Professor NEUBERG.)—Soient A′, B′, C′, D′ les projections des sommets A, B, C, D sur un plan quelconque. Démontrer que les perpendiculaires abaissées des points A′, B′, C′, D′ respectivement sur les faces BCD, CDA, DAB, ABC du tétraèdre sont quatre génératrices d'un même hyperboloïde.

Solution by Professor NANSON.

Let $\alpha, \beta, \gamma, \delta$ be the perpendiculars from A′, B′, C′, D′ on the faces a, b, c, d of ABCD. Draw through D′ planes x, y, z parallel to a, b, c. Then the planes D′α, D′β, D′γ, being perpendicular to a, b, c, are also perpendicular to x, y, z, and are therefore collinear (these planes pass also through the edges of the trihedral angle formed by a, b, c) because the three perpendiculars of a spherical triangle are concurrent. Thus the four $\alpha, \beta, \gamma, \delta$ all intersect a line through D′. Similarly they all intersect lines through A′, B′, C′, and are therefore generators of the same hyperboloid.

16212. (W. J. GREENSTREET, M.A.)—Find the locus of the feet of the perpendiculars from the origin to the normals to Bernoulli's lemniscate.

Solution by Professor W. H. H. HUDSON, M.A.

The solution printed is wrong. There seems to be a confusion as to the meaning of a; a is not defined.

If a be the angle vector of the point at which the normal is drawn, referred to the centre and transverse axis, length $2a$, of the rectangular hyperbola of which the lemniscate is the inverse and the pedal with respect to the centre, r, θ co-ordinates of the foot of the perpendicular, we may deduce from the geometry of the figure

$$r^2 = a^2 \sin^2 2a \cos 2a, \quad a - \theta = \tfrac{1}{2}\pi - 2a;$$

whence the equation of the locus is

$$r^2 = a^2 \sin^2 \tfrac{1}{3}(\pi + 2\theta) \cos \tfrac{1}{3}(\pi + 2\theta).$$

16192. (G. BOOL, B.A.)—Find the solution of

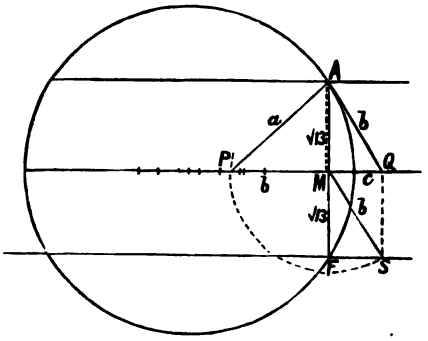
$$a^2 - b^2 = b^2 - c^2 = 13.$$

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(II.) *by* W. D. ROBERTS.

(I.) $a^2 - b^2 = 13$, $b^2 - c^2 = 13$ give $a^2 - c^2 = 26$; whence $a = \mp 7\sqrt{2}$, $b = 6\sqrt{2}$, and $b^2 = \frac{1}{2}(a^2 + c^2) = 85$, $b = \pm\sqrt{85}$.

Similarly, the solutions may be found of $a^2 - b^2 = b^2 - c^2 = \omega$ (an odd number), viz., $a = \pm\frac{1}{2}(\omega + 1)\sqrt{2}$, $c = \pm\frac{1}{2}(\omega - 1)\sqrt{2}$, $b = \pm\sqrt{[\frac{1}{2}(\omega^2 + 1)]}$. Also, if ω be composite, $= xy = \alpha^2 - \gamma^2$, suppose, then every such resolution of ω into a difference of squares (α, γ) gives solutions $a = \pm\alpha\sqrt{2}$, $c = \pm\gamma\sqrt{2}$, $b = \pm(\alpha^2 + \gamma^2)^{\frac{1}{2}}$.

(II.) We have $a > b > c$ numerically. By Euc. II. 14, draw $AM = \sqrt{13}$; produce AM to F so that $MF = \sqrt{13}$. Through A, M, and F draw parallels. Take any point P on the mid-parallel so that $MP > AM$. With centre M and radius PM draw an arc cutting the



lower parallel at S; draw SQ parallel to AF; join AQ. Clearly $AP^2 - PM^2 = 13$, and $AQ^2 - MQ^2 = 13$; *i.e.*, $MS^2 - MQ^2 = 13$ for

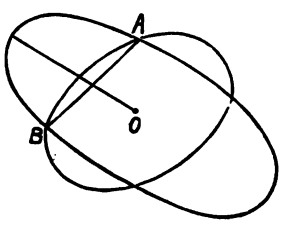
AQ = MS; *i.e.*, $PM^2-MQ^2 = 13$ for MS = PM; therefore $AP^2-PM^2 = PM^2-MQ^2 = 13$, and, if AP = a, PM = b, MQ = c, then $a^2-b^2 = b^2-c^2 = 13$. The locus of P being the mid-parallel left from M, starting from a point distant $\sqrt{13}$ from M.

[NOTE.—The EDITOR remarks that evidently this gives only half of the locus, the other half being derivable by considerations of symmetry and from the second intersections of the variable circle with FS.]

16229. (Professor NANSON.)—Prove geometrically that two concentric conics have a pair of conjugate diameters in common, unless their asymptotes are real and interlaced.

Solution by M. T. NARANIENGAR, M.A.

If the conics cut in four real points, then, obviously, the common conjugate diameters are those bisecting the common chords. If the conics do not cut at all, it is possible to find a conic similar to one which shall cut the other or its conjugate in four real points. Hence also the problem is solved. If they cut in two real points only, their asymptotes are real and interlaced, and then the problem is not possible.



16213. (R. N. APTE.)—The normals at P, Q of a parabola are at right angles, and from their point of intersection the third normal is drawn and meets the curve in R. Find the least distance of the centroid of the triangle PQR from the vertex. Also find the least distance from the vertex of the centroid of the triangle formed by tangents at P, Q, R.

Solutions (I.) *by* W. F. BEARD, M.A., *and others*; (II.) *by* Professor SANJÁNA, M.A.

(I.) Let the co-ordinates of P, Q, R be am_1^2, $2am_1$, ...; then $m_1m_2 = -1$ and $\Sigma m_1 = 0$. Because $\Sigma m_1 = 0$, therefore

$$\Sigma am_1^2 = -2a\Sigma m_2m_3 = -2a\,[m_3(m_1+m_2)+m_1m_2] = 2a\,(1+m_3^2).$$

Thus the co-ordinates of the centroid of PQR are $\frac{1}{3}[2a\,(1+m_3^2)]$, 0; therefore its least distance from the vertex is $\frac{2}{3}a$.

If the tangents at P, Q, R meet at A, B, C, the co-ordinates of A, B, C are am_2m_3, $a\,(m_2+m_3)$, ... Thus the co-ordinates of the centroid of ABC are $-\frac{1}{3}\,[a\,(1+m_3^2)]$, 0 as above, and therefore the least distance of the centroid from the vertex is $\frac{1}{3}a$.

(II.) Let $y = mx - 2am - am^3$ be any normal; then its foot i $(am^2, -2am)$ on the curve $y^2 = 4ax$. If the normal goes through (h, k), its three possible directions are connected by the relations

$$m_1 + m_2 + m_3 = 0, \quad m_1m_2 + m_2m_3 + m_3m_1 = (2a-h)/a, \quad m_1m_2m_3 = -k/a.$$

When two of these are at right angles (h, k) moves on $y^2 = a(x-3a)$; so that the least value of h is $3a$. Hence the distance of the centroid of PQR (which lies on the axis), being $\frac{1}{3}[a(\Sigma m_1^2)]$, is $\frac{1}{3}(2h-4a)$, and has the minimum value $\frac{2}{3}a$. The distance of the centroid of the tangent triangle (which also lies on the axis) being $\frac{1}{3}[a(\Sigma m_1m_2)]$, is $\frac{1}{3}(2a-h)$, and has the numerical minimum $\frac{1}{3}(-a)$; this, however, is really a maximum.

16203. (R. F. Scott, M.A.)—Find a solution of the system of n equations

$$x_1 - a_1 = x_2 - a_2 = \ldots = x_n - a_n,$$

$$\begin{vmatrix} x_1, & x_2, & \ldots, & x_n \\ x_n, & x_1, & \ldots, & x_{n-1} \\ \ldots & & \ldots & \ldots \\ x_2, & x_3, & \ldots, & x_1 \end{vmatrix} = 0.$$

Solution by A. M. Nesbitt, M.A., *and many others.*

A solution is evidently got from the $n-1$ equations

$$x_1 - a_1 = \ldots = x_n - a_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the linear factor Σx from the determinant, equated to zero. If each member of (1) be λ, $\Sigma x = \Sigma a + n\lambda$; so that $\lambda = -\Sigma a/n$. Hence $x_r = a_r - \Sigma a/n$. This is all that is asked for: in a particular case (say $n = 3$) we have, if $\Sigma x \neq 0$, $\Sigma x_1^2 = \Sigma x_2x_3$, or

$$\Sigma a_1^2 + \lambda\Sigma(2a_1 - a_2 - a_3) = \Sigma a_2a_3 : \text{ i.e., } \Sigma a_1^2 = \Sigma a_2a_3;$$

so that no other solution is possible, and it is easy to see that there is only one finite root for λ if $n = 4$.

16218. (Professor Lauvernay.)—Construire un triangle, connaissant les rapports a/b, b/c et la somme $m_a + m_b + m_c$ des médianes.

Solution by J. Blaikie, M.A.

Let $a/b = k$, $b/c = l$. By a known theorem, we have

$$4m_a^2 = 2b + 2c^2 - a^2 = c^2(2l^2 + 2 - k^2l^2) = L^2c^2,$$
$$4m_b^2 = 2c^2 + 2a^2 - b^2 = c^2(2 + 2k^2l^2 - l^2) = M^2c^2,$$
$$4m_c^2 = 2a + 2b^2 - c^2 = c^2(2k^2l^2 + 2l^2 - 1) = N^2c^2,$$

therefore $$2(m_a + m_b + m_c) = (L + M + N)c.$$

Thus c is known, and hence b and a.

16176. (Professor LAUVERNAY.)—Construire la courbe
$\rho^2 - 4\rho \cos \frac{1}{3}\omega + 1 = 0$.

Remarks on the Solution by HENRY RIDDELL, M.E., *by* C. DE POLIGNAC.

From the data, the curve should have a multiple point on the axis, not the loop drawn in the figure illustrating the Solution. For it follows from Mr. Riddell's ingenious construction that there is a (1, 1)-point correspondence between the fundamental circle and the curve: *e.g.*, Fig. 1, the line $Om'M'$ corresponds to OmM where $\angle M'OX = 3\angle MOX$; the point m' corresponds to m, M' to M. Hence when

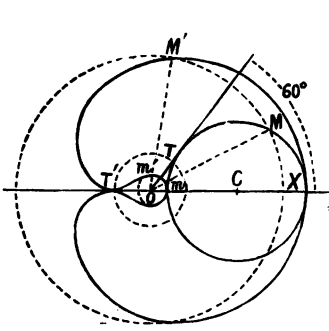


FIG. 1.

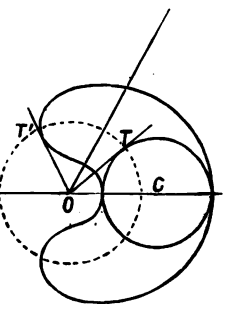


FIG. 2.

the line drawn through the origin touches the circle the corresponding line touches the curve. Now the points of the curve which lie on the axis are given by the values $\omega = 0$ and $\omega = 60°$ as $3.60° = 180°$. Hence in the latter case these points correspond to the points of intersection with the circle of the line $y = x \tan 60°$, *i.e.*, $y = x\sqrt{3}$, and, as this line touches the circle $\rho^2 - 4\rho \cos \omega + 1$ or $x^2 + y^2 - 4x + 1$, the points on the axis coincide and the curve has the shape drawn in Fig. 1. The radius of the circle $\rho^2 - 4\rho \cos \omega + 1 = 0$ is $\sqrt{3}$. Suppose now we take a circle with a radius $< \sqrt{3}$, say $\sqrt{(3-\delta)}$, its centre remaining at the distance 2 from the origin. Its equations will be respectively

$(x-2)^2 + y^2 = 3 - \delta$ or $x^2 + y^2 - 4x + 1 + \delta = 0$; $\rho^2 - 4\rho \cos \omega + 1 + \delta = 0$.

The graphic construction remaining the same as before, a curve $\rho' = f(\theta)$ will be obtained such that to each point (ρ', θ) of the curve corresponds, on the circle, a point (ρ, ω) with $\rho = \rho'$, $\omega = \frac{1}{3}\theta$. The equation of the curve is then $\rho^2 - 4\rho \cos \frac{1}{3}\theta + 1 + \delta = 0$. The points on the axis still correspond to $\omega = 0$, $\theta = 0$; $\omega = 60°$, $\theta = 180°$, but the line $y = x \tan 60°$ or $y = x\sqrt{3}$ meets here the circle $x^2 + y^2 - 4x + 1 + \delta = 0$ in two imaginary points and the tangent to the circle from the origin makes with the axis an angle less than 60°. Hence the corresponding tangent to the curve from the origin makes with the axis an angle less than 180°, as illustrated in Fig. 2.

Finally, if the radius of the circle is greater than $\sqrt{3}$, say $\sqrt{(3+\delta)}$,

then the line $y = x \tan 60°$ meets the circle in two real points to which correspond, on the axis, two real points of the curve, which has then a loop as in the figure drawn in *The Educational Times*.

NOTE.—In response to the above remarks Mr. RIDDELL writes:—"Prince C. de Polignac properly points out that in the figure given the origin was taken too close to the centre of the auxiliary circle, thus forming a loop where there should have been a quadruple point.

"The method was framed for the general case $\rho^2 - a\rho \cos \omega/N + C = 0$, and the figure was not checked against the special equation until too late to alter. There will be only two real points of intersection of this curve with the axis, or the curve will also be tangent with a multiple point according as $\sqrt{(a^2/4 - C)}$ is less than or equal to $\frac{1}{2}a \sin \pi/N$. The total number of points of intersection, real and imaginary, depends, of course, on N. The series of curves where N is fractional is exceedingly interesting and easily constructed with approximate accuracy."

16097. (W. J. GREENSTREET, M.A.)—Show that

$$\frac{b^2}{ab+a-} \; \frac{ab}{a+b+ab-} \; \frac{ab}{a+b+ab-} \cdots$$

to n quotients, identical after the first, is equal to $\frac{b}{a+} \; \frac{a}{b+} \; \frac{b}{a+} \; \frac{a}{b+} \cdots$ to $2n$ quotients.

Solution by C. M. ROSS.

If $\frac{p_n}{q_n}$ is the n-th convergent,

$$p_n = (a+b+ab)\, p_{n-1} - ab\, p_{n-2}, \quad q_n = (a+b+ab)\, q_{n-1} - ab\, q_{n-2} \ldots (1, 2).$$

From (1) it is seen that the numerators form a recurring series, any three consecutive terms being connected by

$$p_n - (a+b+ab)\, p_{n-1} + ab\, p_{n-2}.$$

If S is the sum of this series,

$$S = \{p_1 + [p_2 - p_1 (a+b+ab)]\, x\} / [1 - (a+b+ab)\, x + abx^2].$$

But the first two convergents are $\frac{b^2}{ab+a}$, $\frac{b^2 (a+b+ab)}{a[a(1+b)^2 + b^2]}$; therefore

$$S = \frac{b^2}{1 - (a+b+ab)\, x + abx^2}$$

$$= \frac{b^2}{a\beta - ab} \left[\frac{a}{a} \left(1 - \frac{a}{a} x\right)^{-1} - \frac{b}{\beta} \left(1 - \frac{b}{\beta} x\right)^{-1} \right],$$

where $$a\beta = 1, \quad ab + a\beta = a + b + ab;$$

whence p_n = coefficient of x^{n-1} on the dexter

$$= \frac{b^2}{a\beta - ab} \left[\frac{a}{a} \frac{a^{n-1}}{a^{n-1}} - \frac{b}{\beta} \frac{b^{n-1}}{\beta^{n-1}} \right] = \frac{b^2}{a\beta - ab} \left[\left(\frac{a}{a}\right)^n - \left(\frac{b}{\beta}\right)^n \right].$$

Similarly,

$$\begin{aligned} S' &= \{q_1 + [q_2 - q_1(a+b+ab)]\,x\}/[1-(a+b+ab)\,x+abx^2] \\ &= (ab+a-abx)/[1-(a+b+ab)\,x+abx^2] \\ &= \frac{1}{a\beta-ab}\left[\frac{a}{\alpha}(a+ab-ab)\left(1-\frac{a}{\alpha}x\right)^{-1}\right. \\ &\qquad \left. -\frac{b}{\beta}(a+ab-a\beta)\left(1-\frac{b}{\beta}x\right)^{-1}\right]. \end{aligned}$$

Now q_n = coefficient of x^{n-1} on the dexter

$$= \frac{1}{a\beta-ab}\left[(a\beta-b)\left(\frac{a}{\alpha}\right)^n-(ab-b)\left(\frac{b}{\beta}\right)^n\right];$$

therefore
$$\frac{p_n}{q_n} = \frac{b^2\left[\left(\frac{a}{\alpha}\right)^n-\left(\frac{b}{\beta}\right)\right]}{(a\beta-b)\left(\frac{a}{\alpha}\right)^n-(ab-b)\left(\frac{b}{\beta}\right)^n}.$$

As before, it will be found that p'_{2n} is the coefficient of x^{2n-1} in the expansion of

$$\begin{aligned} S_1 &= \frac{b+b^2x-bax^2}{1-(a+b+ab)\,x^2+abx^4} \\ &= \frac{b}{2(a\beta-ab)}\left[\frac{a-a\alpha+b\sqrt{(a\alpha)}}{\alpha}\left(1-x\sqrt{\frac{a}{\alpha}}\right)^{-1}\right. \\ &\qquad +\frac{a-a\alpha-b\sqrt{(a\alpha)}}{\alpha}\left(1+x\sqrt{\frac{a}{\alpha}}\right)^{-1} \\ &\qquad +\frac{a\beta-b-b\sqrt{(b\beta)}}{\beta}\left(1-x\sqrt{\frac{b}{\beta}}\right)^{-1} \\ &\qquad \left.+\frac{a\beta-b+b\sqrt{(b\beta)}}{\beta}\left(1+x\sqrt{\frac{b}{\beta}}\right)^{-1}\right]; \end{aligned}$$

whence

$$\begin{aligned} p'_{2n} &= \frac{b}{2(a\beta-ab)}\left\{\left(\frac{a}{\alpha}\right)^{\frac{1}{2}(2n-1)}\frac{1}{\alpha}[a-a\alpha+b\sqrt{(a\alpha)}-a+a\alpha+b\sqrt{(a\alpha)}]\right. \\ &\qquad \left.+\left(\frac{b}{\beta}\right)^{\frac{1}{2}(2n-1)}\frac{1}{\beta}[a\beta-b-b\sqrt{(b\beta)}-a\beta+b-b\sqrt{(b\beta)}]\right\} \\ &= \frac{b^2}{a\beta-ab}\left[\left(\frac{a}{\alpha}\right)^{\frac{1}{2}(2n-1)}\frac{a^{\frac{1}{2}}}{\alpha}-\left(\frac{b}{\beta}\right)^{\frac{1}{2}(2n-1)}\frac{b^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right] \\ &= \frac{b^2}{a\beta-ab}\left[\left(\frac{a}{\alpha}\right)^n-\left(\frac{b}{\beta}\right)^n\right]. \end{aligned}$$

Similarly, $q'_{2n} = \frac{1}{a\beta-ab}\left[(a\beta-b)\left(\frac{a}{\alpha}\right)^n-(ab-b)\left(\frac{b}{\beta}\right)^n\right];$

therefore
$$\frac{p'_{2n}}{q'_{2n}} = \frac{b^2\left[\left(\frac{a}{\alpha}\right)^n-\left(\frac{b}{\beta}\right)^n\right]}{(a\beta-b)\left(\frac{a}{\alpha}\right)^n-(ab-b)\left(\frac{b}{\beta}\right)^n};$$

therefore
$$\frac{p_n}{q_n} = \frac{p'_{2n}}{q'_{2n}}.$$

16200. (Professor SANJÁNA, M.A.)—The great circle bisecting perpendicularly the side BC of the spherical triangle ABC meets the great circles bisecting internally and externally the angle A in the points D, E. Prove that D and E lie on the small circle circumscribing ABC.

Remarks by M. T. NARANIENGAR, M.A.

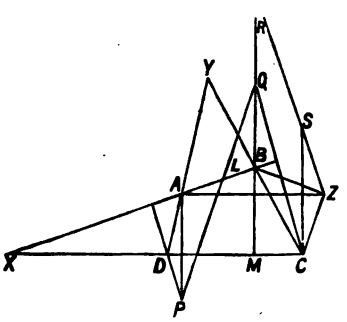
The Question is evidently incorrect and the Solution wrong. The bisectors of the angles A of the plane triangle ABC are not the traces of the bisectors of the angle A of the spherical triangle ABC, and hence the solution fails. The Question proposed makes the spherical excess of the triangle ADE vanish, which is evidently incorrect.

16183. (Professor R. W. GENESE, M.A.)—Inward forces act at the mid-points of the sides of any plane quadrilateral at right angles and proportional to the opposite sides. Prove (1) that they balance, (2) that the resultant of two opposite forces acts along the directrix of the parabola inscribed in the quadrilateral.

Note, suggested by Mr. RIDDELL'S *Extension of the Problem, by* S. T. SHOVELTON, M.A.

Mr. Riddell's extension of Professor Genese's problem suggests a proof which is interesting from a geometrical point of view.

Let P and Q be the orthocentres of the triangles XAD and XBC. Then PQ is the directrix of the parabola inscribed in ABCD.



Let L and M be such that AL/LB = DM/MC. Then, by parallels, it is seen that the perpendiculars from L on DC and from M on AB meet PQ in the same point.

Let ADCZ be a parallelogram. Then the resultant of forces at L and M, perpendicular and proportional to the opposite sides, is a force perpendicular and proportional to BZ.

To prove that this resultant acts along PQ—*i.e.*, to prove PQ perpendicular to BZ.

Let ZR be perpendicular to AB and meet BQ in R. Then R is the orthocentre of ABZ. Therefore AR is perpendicular to BZ. If CS be perpendicular to AZ, it is seen that AP = CS = QR. Therefore AR is parallel to PQ.

Therefore PQ is perpendicular to BZ (or, more shortly, by Brianchon's Theorem, BZ is parallel to axis of parabola and therefore perpendicular to PQ).

Therefore the resultant acts along the directrix PQ.

Similarly, if N and O divide BC and AD in any (the same) ratio, the resultant of the forces through N and O acts along the directrix, and both parts of the proposition are proved.

16251. (W. F. BEARD, M.A.)—If a system of conics have a common chord, then the polars of any point on the common chord, with regard to the conics, are concurrent.

Solution by S. NARAYANAN, R. J. WHITAKER, B.A., *and the* PROPOSER.

If O be any point in the common chord AB of a system of conics, the polars of O with respect to the conics pass through a common point in AB, viz., the harmonic conjugate of O with respect to A, B.

16209. (J. J. BARNIVILLE, B.A., I.C.S.)—Prove that

$$\frac{1}{1.3}+\frac{1.2}{3.7}+\frac{3.2^2}{7.17}+\frac{7.2^3}{17.41}+\ldots=1;$$

$$\frac{1}{1.3}+\frac{2.2}{3.8}+\frac{5.2^2}{8.21}+\frac{13.2^3}{21.55}+\ldots=1.$$

Solutions by Prof. SANJÁNA, M.A., C. M. ROSS, *and others.*

(1) In 1, 1, 3, 7, 17, ..., we have $u_n+2u_{n+1}=u_{n+2}$; hence general term of given series $=\dfrac{u_n 2^{n-1}}{u_{n+1}u_{n+2}}=\dfrac{(u_{n+2}-2u_{n+1})\,2^{n-1}}{u_{n+1}u_{n+2}}=\dfrac{2^{n-1}}{u_{n+1}}-\dfrac{2^n}{u_{n+2}}$.

Thus the sum $=(\frac{1}{1}-\frac{2}{3})+(\frac{2}{3}-2^2/7)+(2^2/7-2^3/17)\ldots=1$, as the limit of $2^n/u_{n+1}$ may be proved to be zero.

(2) In 0, 1, 3, 8, 21, ..., we have $u_{n+2}=3u_{n+1}-u_n$; also 1, 2, 5, 13, ..., is the series of first differences. Thus the general term

$$=\frac{(u_n-u_{n-1})\,2^{n-1}}{u_nu_{n+1}}=\frac{(u_{n+1}-2u_n)\,2^{n-1}}{u_nu_{n+1}}=\frac{2^{n-1}}{u_n}-\frac{2^n}{u_{n+1}}.$$

Therefore the sum $=(\frac{1}{1}-\frac{2}{3})+(\frac{2}{3}-2^2/8)+(2^2/8-2^3/21)\ldots=1$, for the same reason as above.

16210. (A. M. NESBITT, M.A.)—Prove from first principles that the locus of a point whence a straight line can be drawn to intersect

$(x-a_r)/l_r = (y-\beta_r)/m_r = (z-\gamma_r)/n_r$ $(r = 1, 2, 3)$ is $|P_1, Q_2, R_3| = 0$, where P_k, Q_k, R_k are the co-factors of p_k, q_k, r_k in the determinant

$$\begin{vmatrix} p_k, & q_k, & r_k \\ l_k, & m_k, & n_k \\ x-a_k, & y-\beta_k, & z-\gamma_k \end{vmatrix}.$$

[Note that the locus is a quadric and that the three given lines are obviously generators.]

PROPOSER'S *close Discussion of the Question.*

Let (a_r, β_r, γ_r) be called the point (r); (a, β, γ) the point P. Now the plane through P to meet line (2) has for equation

$$\begin{vmatrix} x, & y, & z, & 1 \\ a, & \beta, & \gamma, & 1 \\ a_2, & \beta_2, & \gamma_2, & 1 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = 0,$$

and, writing 3 for 2, we get a similar equation for the plane through P to meet line (3). These two planes intersect in the line through P which meets lines (2) and (3), and this line, by hypothesis, intersects line (1), whose equations are

$$m_1x - l_1y - (m_1a_1 - l_1\beta_1) = 0, \qquad n_1x - l_1z - (n_1a_1 - l_1\gamma_1) = 0.$$

Thus the four planes just written have a common point: so that

$$\begin{vmatrix} m_1, & l_1, & 0, & m_1a_1 - l_1\beta_1 \\ n_1, & 0, & l_1, & n_1a_1 - l_1\gamma_1 \\ \begin{vmatrix} \beta, & \gamma, & 1 \\ \beta_2, & \gamma_2, & 1 \\ m_2, & n_2, & 0 \end{vmatrix}, & \begin{vmatrix} a, & \gamma, & 1 \\ a_2, & \gamma_2, & 1 \\ l_2, & n_2, & 0 \end{vmatrix}, & \begin{vmatrix} a, & \beta, & 1 \\ a_2, & \beta_2, & 1 \\ l_2, & m_2, & 0 \end{vmatrix}, & |a, \beta_2, n_2| \\ \begin{vmatrix} \beta, & \gamma, & 1 \\ \beta_3, & \gamma_3, & 1 \\ m_3, & n_3, & 0 \end{vmatrix}, & \begin{vmatrix} a, & \gamma, & 1 \\ a_3, & \gamma_3, & 1 \\ l_3, & n_3, & 0 \end{vmatrix}, & \begin{vmatrix} a, & \beta, & 1 \\ a_3, & \beta_3, & 1 \\ l_3, & m_3, & 0 \end{vmatrix}, & |a, \beta_3, n_3| \end{vmatrix} = 0.$$

If now we multiply the first column by a, the second by $-\beta$, the third by $-\gamma$, and the fourth by -1, and add for a new fourth column, and, further, denote $a-a_r$ by x_r, $\beta-\beta_r$ by y_r, $\gamma-\gamma_r$ by z_r, the four elements in the new fourth column are $m_1x_1 - l_1y_1$, $n_1x_1 - l_1z_1$, 0, 0; while those in the third row may be written $m_2z_2 - n_2y_2$, $l_2z_2 - n_2x_2$, $l_2y_2 - m_2x_2$, 0; and similarly for the fourth row.

Change the sign of the second column and put the fourth first; thus the determinant becomes

$$\begin{vmatrix} m_1x_1 - l_1y_1, & m_1, & -l_1, & 0 \\ n_1x_1 - l_1z_1, & n_1, & 0, & -l_1 \\ 0, & m_2z_2 - n_2y_2, & n_2x_2 - l_2z_2, & l_2y_2 - m_2x_2 \\ 0, & m_3z_3 - n_3y_3, & n_3x_3 - l_3z_3, & l_3y_3 - m_3x_3 \end{vmatrix} = 0.$$

Make a new second column by multiplying the second and third and fourth columns by x_2, y_2, z_2 and adding: thus

$$\begin{vmatrix} m_1x_1-l_1y_1, & m_1x_2-l_1y_2, & -l_2, & 0 \\ n_1x_1-l_1z_1, & n_1x_2-l_1z_2, & 0, & -l_1 \\ 0, & 0, & n_2x_2-l_2z_2, & l_2y_2-m_2x_2 \\ 0, & |\, x_2, m_3, z_3 \,|, & n_2x_3-l_2z_3, & l_2y_3-m_2x_3 \end{vmatrix} = 0.$$

Put $L_r = m_rz_r-n_ry_r$, $M_r = n_rx_r-l_rz_r$, ...; $\lambda_r = m_1z_r-n_1y_r$, ... Thus

$$\begin{vmatrix} 0, & 0, & M_3, & N_2 \\ -\nu_1, & -\nu_2, & -l_1, & 0 \\ \mu_1, & \mu_2, & 0, & -l_1 \\ 0, & |\, x_2, m_3, z_3 \,|, & M_3, & N_3 \end{vmatrix} = 0$$

or $$M_2 \begin{vmatrix} \nu_1, & \nu_2, & 0 \\ \mu_1, & \mu_2 & -l_1 \\ 0, & |\, x_2, m_3, z_3 \,|, & N_3 \end{vmatrix} = N_2 \begin{vmatrix} \nu_1, & \nu_2, & l_1 \\ \mu_1, & \mu_2, & 0 \\ 0, & |\, x_2, m_3, z_3 \,|, & M_3 \end{vmatrix}$$

or $$M_2 \left\{ l_1, \nu_1, D+N_3 \begin{vmatrix} \nu_1, & \nu_2 \\ \mu_1, & \mu_2 \end{vmatrix} \right\} = N_2 \left\{ l_1, \mu_1, D+M_3 \begin{vmatrix} \nu_1, & \nu_2 \\ \mu_1, & \mu_2 \end{vmatrix} \right\},$$

D being written for $|\, x_2, m_3, z_3 \,|$. Thus

$$l_1 D \begin{vmatrix} M_2, & N_2 \\ \mu_1, & \nu_1 \end{vmatrix} = \begin{vmatrix} \mu_1, & \mu_2 \\ \nu_1, & \nu_2 \end{vmatrix} \begin{vmatrix} M_2, & M_3 \\ N_2, & N_3 \end{vmatrix}.$$

Now, if large letters be co-factors of corresponding small letters in

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ x_2, & y_2, & z_2 \\ x_1, & y_1, & z_1 \end{vmatrix} = \Delta,$$

$$\mu_1 = -Y_2, \quad \mu_2 = -Y_1, \quad \nu_1 = -Z_2, \quad \nu_2 = -Z_1;$$

so that $$\begin{vmatrix} \mu_1, & \mu_2 \\ \nu_1, & \nu_2 \end{vmatrix} = \begin{vmatrix} Y_2, & Z_2 \\ Y_1, & Z_1 \end{vmatrix} = l_1, \Delta.$$

Thus $$D \begin{vmatrix} M_2, & N_2 \\ \mu_1, & \nu_1 \end{vmatrix} = \Delta \begin{vmatrix} M_2, & M_3 \\ N_2, & N_3 \end{vmatrix}$$

or $$M_2(\nu_1 D - N_3\Delta) = N_2(\mu_1 D - M_3\Delta).$$

But $$D = x_2L_3 + y_2M_3 + z_2N_3, \quad \Delta = x_2\lambda_1 + y_2\mu_1 + z_2\nu_1;$$

therefore $$\nu_1 D - N_3\Delta = x_2(\nu_1L_3 - \lambda_1N_3) + y_2(\nu_1M_3 - \mu_1N_3),$$
$$\mu D - M_3\Delta = x_2(\mu_1L_3 - \lambda_1M_3) + z_2(\mu_1N_3 - \nu_1M_3).$$

Thus

$$-x_2[M_2(\lambda_1N_3 - \nu_1L_3) - N_2(\lambda_1M_3 - \mu_1L_3)] + (N_2z_2 + M_2y_2) \begin{vmatrix} M_3, & N_3 \\ \mu_1, & \nu_1 \end{vmatrix} = 0.$$

But $$L_2x_2 + M_2y_2 + N_2z_2 = 0;$$

therefore, dividing by x_2,

$$M_2 \begin{vmatrix} \lambda_1, & \nu_1 \\ L_3, & N_3 \end{vmatrix} + N_2 \begin{vmatrix} \mu_1, & \lambda_1 \\ M_3, & L_3 \end{vmatrix} + L_2 \begin{vmatrix} \nu_1, & \mu_1 \\ N_3, & M_3 \end{vmatrix} = 0,$$

i.e., $$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix} = 0.$$

Thus the surface has for equation

$$\begin{vmatrix} m_1(\gamma-\gamma_1)-n_1(\beta-\beta_1), & n_1(\alpha-\alpha_1)-l_1(\gamma-\gamma_1), & l_1(\beta-\beta_1)-m_1(\alpha-\alpha_1) \\ m_2(\gamma-\gamma_2)-n_2(\beta-\beta_2), & \ldots, & \ldots \\ m_3(\gamma-\gamma_3)-n_3(\beta-\beta_3), & \ldots, & \ldots \end{vmatrix}$$

This is of the second degree, since terms of the third degree vanish, for

$$\alpha(m_r\gamma-n_r\beta)+\beta(n_r\alpha-l_r\gamma)+\gamma(l_r\beta-m_r\alpha)=0,$$

and, moreover, it plainly passes through any point upon each of the lines (1), (2), (3). It may be written $|\,P_1, Q_2, R_3\,| = 0$, when P_k, Q_k, R_k are co-factors of p_k, q_k, r_k in the determinant

$$\begin{vmatrix} p_k, & q_k, & r_k \\ l_k, & m_k, & n_k \\ \alpha-\alpha_k, & \beta-\beta_k, & \gamma-\gamma_k \end{vmatrix}.$$

16190. (R. W. D. Christie.)—If $N = AB$, all integral, prove that the small factor $A = \frac{1}{2}\{[4BN+(a-bc)^2]^{\frac{1}{2}}-a+bc\}/2b$ and the large factor $B = \frac{1}{2}\{[4bN+(a-bc)^2]^{\frac{1}{2}}+a-bc\}$. Also (2) $(a-bc)^2 = (B-bA)^2$, all the letters being integral: *e.g.*, $N = 257821$. Then one set of values (out of many) is $b = 2$, $c = 10$, $a = 69$; or, again, $b = 2$, $c = 8$, $a = 65$. It is required to find the small letters.

Proposer's *Solution*.

(I.) $N = (a+bx)(c+x) = AB$

$= f(x) = bx^2+(a+bc)x+ac$, $= b(x-c)^2+(a+3bc)(x-c)+f(c)$ (i., ii.),

$= b(x+c)^2+(a-bc)(x+c)+f(-c)$, $= bA^2+(a-bc)A+0$ ... (iii., iv.);

thus $N = A(Ab+a-bc)$ and $B-Ab = a-bc$, which proves (2) in the Question.

(II.) Also $(N-ac)/b = x^2+[(a+bc)/b]\,x$.

Completing the quadratic and rooting,

$$x = \{[4bN+(a-bc)^2]^{\frac{1}{2}}-(a+bc)\}/2b \quad \ldots\ldots\ldots\ldots \text{(v.)};$$

whence $$A = c+x = \{[4bN+(a-bc)^2]^{\frac{1}{2}}-(a+bc)\}/2b \quad \ldots\ldots\ldots \text{(vi.)},$$

$$B = a+bx = \tfrac{1}{2}\{[4bN+(a-bc)^2]^{\frac{1}{2}}+a-bc\} \quad \ldots\ldots\ldots \text{(vii.)},$$

proving the theorem.

(III.) Now $f(-c) = 0$; *i.e.*, $bc^2-(a+bc)c+ac = 0$ ..... (viii.).

(IV.) Now, as b is the integral measure of A into B, one factor into another, it will usually be very small. Let us fix (*e.g.*) the successive values of b at 2 and 3. Then (i.), (ii.), (iii.), (iv.) become

$N = 2x^2+x(a+2c)+ac$, $= 2(x-c)^2+(a+6c)(x-c)+f(c)$ ... (i., ii.),

$= 2(x+c)^2+(a-2c)(x+c)+f(-c)$, $= 2A^2+(a-2c)A+0$ (iii., iv.),

and, similarly, $= 3A^2+(a-3c)A+0$ ..... (iv.A).

Thus, with b fixed and manœuvring with a and c [*v.* equations (i.) and (v.)]. In order to get the same A we have, *e.g.*: Let $b = 2$, $a = 69$, $c = 10$; $b_1 = 3$. $a = 2$, $c = 100$;

$$N = 257821 = 2.347^2 + 49.347 = 3.347^2 - 298.347.$$

The factors are $2 \times 347 + 49$ or $3 \times 247 - 298$ and 347.

The theorem, though not primarily intended as such, affords useful aid towards the factorization of a large N.

16242. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize completely (into prime factors) $N = (3^{54} + 2^{54})$.

Solution by the PROPOSER.

Let $N = (x^6 + y^6)/(x^2 + y^2) = (x^2 + 3xy + y^2)^2 - 6xy\,(x + y)^2$
$= P^2 - Q^2$ when $6xy = \square$.

Let $x = 3\xi^2$, $y = 2\eta^2$; then $6xy = (6\xi\eta)^2$. Then N is a Sext-Aurifeuillian, algebraically resolvable into the factors $L = P - Q$, $M = P + Q$ when

$$P = x^2 + 3xy + y^2, \quad M = 6\xi\eta\,(x + y).$$

Ex.—Let $N = 3^{54} + 2^{54} = N_9 N_3 N_1 N_0$ when

$$N_9 = (3^{54} + 2^{54})/(3^{18} + 2^{18}), \quad N_3 = (3^{18} + 2^{18})/(3^6 + 2^6),$$
$$N_1 = (3^6 + 2^6)/(3^2 + 2^2), \quad N_0 = (3^2 + 2^2) = 13.$$

Here N_1, N_3, N_9 are each Sext-Aurifeuillians, and may be resolved by above rules. But $N_1 = 61$. Let $N_3 = L_3 M_3$, $N_9 = L_9 M_9$.

For N_3 take $\xi = 3$, $\eta = 2$; then $x = 3^3$, $\eta = 2^3$. Hence

$$P_3 = 1441, \quad Q_3 = 1260; \quad L_3 = 181, \quad M_3 = 2701 = 37 \cdot 73.$$

For N_9 take $\xi = 3^4$, $\eta = 2^4$; then $x = 3^9$, $y = 2^9$;

$$P_9 = (3^{18} + 3 \cdot 6^9 + 2^{18}), \quad Q = 6 \cdot 6^4\,(3^9 + 2^9),$$
$$L_9 = 260879401 = 109 \cdot 2393389, \quad M_9 = 574952041;$$

the character of M_9 is not known: it contains no divisors < 10000, and its divisors (if any) must be primes of forms

$$p = 108\varpi + 1 = a^2 + b^2 = A^2 + 3B^2 = A'^2 - 3B'^2.$$

15254. (Professor SANJÁNA, M.A.)—On the sides AB, AC ($c > b$) of the triangle of reference similar segments of circles are described towards BC. Prove that the locus of the other point of intersection of these circles is the nodal circum-cubic

$$\alpha(\beta^2 - \gamma^2) = 2\beta\gamma\,(\gamma \cos C - \beta \cos B).$$

Show that the tangents to the cubic at B and C and the line joining A

to the circum-centre of the triangle all meet the cubic at the same point, and that the satellite of BC touches the curve, and is given by

$$a(\cos^2 B - \cos^2 C) = 2(\beta \cos^3 C - \gamma \cos^3 B).$$

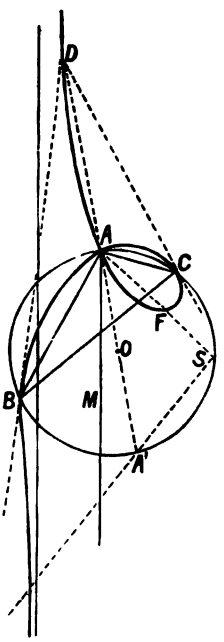
Also find the point of inflexion on the cubic.

Solution by C. E. YOUNGMAN, M.A.

Let the two circles meet at P; then the angles APB, APC are equal or supplementary; so that A is equidistant from BP and CP, and therefore area APB : area ACP = BP : CP; hence the equation of the curve, $b\beta . BP = c\gamma . CP$. Squared, this gives

$$\beta^2(\gamma^2 + 2\gamma a \cos\beta + a^2) = \gamma^2(a^2 + 2a\beta \cos C + \beta^2),$$

which reduces directly to the form given in the Question. Let P coincide with B; then A is equidistant from BB (the tangent at B) and CB; similarly, A is equidistant from BC and CC; therefore, if BB and CC meet at D, A is an in- or ex-centre of DBC. Hence D is a point on the curve, and also AD goes through the circum-centre of ABC. The satellite of BC must pass twice through D, and therefore touches the curve there.



Similar segments cut their bases at equal angles; therefore, by inversion from A, the segments APB, APC become lines pb, pc such that Ap subtends equal or supplementary angles at b and c; therefore p describes a rectangular hyperbola Abc with bc for a diameter. Let s be the centre of this hyperbola, pp' another diameter, and q another point on the curve; then the circles Apq, A$p'q$ are always equal, and consequently (if we make q coincide first with p and then with p') the circles App, A$p'p'$, and App' are three equal circles meeting at a point. The first two therefore meet again at r, the orthocentre of App' — on the hyperbola. Hence, if the circle of curvature at p goes through A, r must coincide with p; that is, App' must be a right angle, and p must lie on the circle which has As for diameter. In the cubic, then, points of inflexion must lie on the line drawn through a certain point S at right angles to AS. Since As is a median of Abc, AS is a symmedian chord of the circle ABC; hence the co-ordinates of S are $(-a : 2b : 2c)$. And A', opposite to A on the circle ABC, has co-ordinates $(-1; \sec C : \sec B)$; therefore the equation of SA', the line of inflexion, is $2a + \beta \cos C + \gamma \cos B = 0$.

Substituting from this for a in the equation of the curve, we get

$$\beta^3 \cos C - 3\beta^2\gamma \cos B + 3\beta\gamma^2 \cos C - \gamma^3 \cos B = 0,$$

the real root of which determines the point of inflexion. Incidentally the above inversion proves that any circle through AS cuts the cubic in points P, P′ whose tangents meet at R on the curve, AR being orthogonal to the circle; also, since the hyperbola is determined just as well by A and pp' as by A and bc, the cubic can be derived from APP′ in the same way as from ABC. Hence the triangles APP′ have all the same symmedian AS, the same bisectors of the angle A (the tangents at the node), and therefore the same median line AM. The projection of A on PP′ is a point on the cubic, because there the semicircles on AP, AP′ meet; it is, in fact, the point R′ concyclic with ARS, because AR, AR′ are isogonal in the angle PAP′. Another point on the cubic is F, the mid-point of AS, the inverse of f opposite to A on the hyperbola, and pairing with it is the inverse of A, namely, the point F′ at infinity on AM. The circular points I, J are another pair, and, since I, J, F′ are collinear (and AI, AJ isogonal), the tangents at I and J (the imaginary asymptotes) meet at F, which is therefore the double focus of the cubic. Let FP cut the median at M. Since the figures FAP, FP′A are similar, $\angle FPA = \angle FAP' = \angle PAM$; therefore MP = MA. Hence a construction by points for the curve, given by the Rev. Mr. Jones in the *Ladies' Diary* for 1776: "From the pole F draw any line at pleasure cutting AM in some point M; then, with the centre M and radius MA, sweep an arch cutting the line from the pole in two points on the curve." Thus any chord through the double focus will subtend a right angle at the node. The real asymptote (the subject of the *Diary* question) is parallel to AM, and the same distance from it as F.

The curve is one which often presents itself, under one aspect or another: it was recently studied by Mr. H. L. Trachtenberg (*Math. Gazette*, Vol. III., p. 288) as the locus of points of contact of tangents (or of the feet of normals) from A to conics of the system with foci B and C.

16187. (V. Ramaswami Aiyar, M.A.)—A. and B. play at a game, in which each game is won by A. or B. by an equal chance, for a fixed stake on each game. B. has enough money; while A., having none, poses as if he has enough, too. Show that the probability that A. would be able to maintain his pretension for $2n-1$ games or $2n$ games is

$$\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\ldots\,\frac{2n-1}{2n}.$$

Solution by S. Narayana Aiyar.

[*Lemma.*—$C_{m,n}$ being the number of games in which m gains and n losses can occur, and $J_{m,n}$ denoting the number of orders which fulfil the condition that the losses must never be in excess of the gains

$$J_{m,n} = C_{m,n} - C_{m+1,n-1}.$$

Whitworth's *Choice and Chance*, Fifth Edition, page 115.]

If the wager be a single coin, at the end of every odd game A. must have coins, and therefore he will be able to continue his pretension for the next game, or $A_{2n-1} = A_{2n}$ where A_{2n} denotes the probability that A. continues his pretension to $2n$ games.

At the end of every even game A. should have either no coin or an even number of coins. Let ${}_{2n}F_{2r}$ denote the number of favourable games out of $2n$ games to have $2r$ coins. Then, by the Lemma, by changing m into $n+r$ and n into $n-r$,

$$J_{n+r,\,n-r}, \quad i.e., \quad {}_{2n}F_{2r} = {}^{2n}C_{n-r} - {}^{2n}C_{n\ r-1}.$$

By giving to r in succession the values 1, 2, 3, ..., n, and remembering that ${}_{2n}F_{2n}$ is 1, the number of favourable games for A. to have coins is ${}^{2n}C_{n-1}$. The number of favourable games for A. to have no coin is ${}_{2n}F_0 = {}^{2n}C_n - {}^{2n}C_{n-1}$, and from these results we get games where A. has coins : games where A. has no coin $= n : 1$.

When A. has coins the chance of his continuing the next game is 1, while when A. has no coin his chance is only $\frac{1}{2}$. But the number of games in which he has coins is n times the number of games in which he has no coins. Therefore his total chance of continuing his next game is $(n+\frac{1}{2})/(n+1)$, *i.e.*, $(2n+1)/(2n+2)$. Therefore

$$A_{2n+1} = (2n+1)/(2n+2)\, A_{2n}.$$

But we have already proved that $A_{2n-1} = A_{2n}$, and, remembering that A_2 is $\frac{1}{2}$, we get

$$A_{2n-1} = A_{2n} = (1.3.5.7 \ldots 2n-1)/(2.4.6.8 \ldots 2n).$$

15994. (Hon. G. R. Dick, M.A.)—Quadrics are drawn through seven given points, four of which are taken as the vertices of the tetrahedron of reference. If (x_1, y_1, z_1, w_1), ..., (x_3, y_3, z_3, w_3) are the co-ordinates (quadriplanar or tetrahedral) of the other three given points, and the notation $|\, a_1\, b_2\, c_3\,|$ is employed to denote the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then the co-ordinates of the eighth point through which all the quadrics of the system pass are given by

$$x\, \frac{|\, x_1 w_1,\ y_2 w_2,\ y_3 z_3\,|}{|\, y_1,\ z_2,\ w_3\,|} = \text{similar quantities.}$$

Note by Professor Nanson.

The remark of the Proposer, *Reprint*, Vol. xi., p. 80, calls for a reply. For simplicity, I assume that no four of the given points are in a plane. By a well known theorem in determinants (Sylvester's homoloidal law) the fifteen determinants formable from a 6.4 array (using all the rows every time) all necessarily vanish if three properly chosen ones vanish. In the Solution, pp. 79, 80, the three planes are

properly chosen in the sense above implied, provided the determinants

$$\begin{Vmatrix} x_2x_3 & y_2y_3 & z_2z_3 & w_2w_3 \\ x_3x_1 & y_3y_1 & z_3z_1 & w_3w_1 \\ x_1x_2 & y_1y_2 & z_1z_2 & w_1w_2 \end{Vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

are not all zero. In this event the three planes have a determinate point of intersection and that point of intersection necessarily, by Sylvester's theorem, lies on each of the quadrics and cubics mentioned in the Solution. If, on the other hand, the determinants (A) do all vanish, then the equations given by the Proposer in the Question take the nugatory form $x \,.\, 0 = y \,.\, 0 = z \,.\, 0 = w \,.\, 0$.

11215. (Professor LEMOINE.)—Sur les perpendiculaires MA′, MB′, MC′ abaissées d'un point M sur les côtés du triangle ABC on prend respectivement les points A″, B″, C″ tels que

$$MA' \,.\, MA'' = MB' \,.\, MB'' = MC' \,.\, MC'' = \rho^2.$$

Quel que soit ρ, le triangle A″B″C″ est perspectif avec ABC et le centre de perspective est sur l'hyperbole équilatère ABCM.

Solutions (I.) *by* Professor A. DROZ-FARNY; (II.) *by* Professor SANJÁNA, M.A.

(I.) Décrivons de M comme centre, avec la quantité ρ comme rayon, une circonférence variable; comme $MA' \,.\, MA'' = \rho^2$, le point A″ est le pôle de la droite BC; il en est de même pour B″ et C″. Or (Casey, *A Sequel to Euclid*, Fourth Edition, p. 142): "If two triangles be such that the sides of one are the polars of the vertices of the other, they are in perspective."

Les points A″, B″, C″ décrivent sur les supports MA′, MB′, MC′ des ponctuelles perspectives ayant le point M en commun correspondant à $\rho = 0$. Le point infini de chaque série est de même point homologue correspondant à $\rho = \infty$.

Les trois faisceaux AA″, BB″, CC″ de centres A, B, C sont donc en homographie, et le point d'intersection de trois rayons homologues décrit une conique passant par A, B, C, par le point M commun aux trois ponctuelles, et par l'orthocentre du triangle pour la valeur $\rho = \infty$, car dans ce cas AA″ devient parallèle à MA″, donc hauteur du triangle.

Le lieu cherché est donc bien l'hyperbole équilatère ABCM.

(II.) Let A″A, B″B, C″C cut BC, CA, AB in X, Y, Z respectively; let f, g, h be the co-ordinates of M. Then

$$\frac{\sin BAX}{\sin CAX} = \frac{h + \rho^2 \cos B/f}{g + \rho^2 \cos C/f} = \frac{hf + \rho^2 \cos B}{fg + \rho^2 \cos C};$$

so $$\frac{\sin CBY}{\sin ABY} = \frac{fg + \rho^2 \cos C}{gh + \rho^2 \cos A} \quad \text{and} \quad \frac{\sin ACZ}{\sin BCZ} = \frac{gh + \rho^2 \cos A}{hf + \rho^2 \cos B};$$

hence, as $$\frac{\sin BAX \sin CBY \sin ACZ}{\sin CAX \sin ABY \sin BCZ} = 1,$$

the three lines are concurrent, so that A″B″C″ is in perspective with ABC. Let x, y, z be the co-ordinates of the centre of perspective; then $x(gh+\rho^2\cos A) = y(hf+\rho^2\cos B) = z(fg+\rho^2\cos C)$. Eliminating ρ^2, we get at once

$$xy(hf\cos A - gh\cos B) + yz(fg\cos B - hf\cos C) + zx(gh\cos C - fg\cos A) = 0,$$

which is the rectangular hyperbola through ABCM.

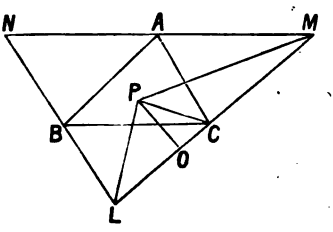
9000. (R. KNOWLES, B.A.)—In any triangle prove that

$$a\cos(B-C) + b\cos(A-C) + c\cos(A-B) = 4/R.$$

[On right-hand side *read* $4\Delta/R$.—ED.]

Solutions (I.) *by* HENRY RIDDELL, M.E.; (II.) *by* R. J. WHITAKER, B.A., Rev. J. C. BOWMAN, *and many others.*

(I.) Draw through the three vertices parallels to opposite sides, thus forming the triangle LMN whose area $= 4\Delta$. Join P, the centre of circum-circle of ABC, to L, C, and M, and draw PO perpendicular to LM. Then is



$$\angle PCB = 90° - A,$$

and $\angle BCL = \angle B$.

Hence we see that

$$\angle PCO = 90° + (B-A), \text{ and } PC = R.$$

$$\sin[90° + (B-A)] = \cos(A-B) \text{ and } PO = R\cos(A-B).$$

But $LC = c$. We have then $Rc\cos(A-B) =$ area LMP, and similarly for the other sides. Hence

$$R[c\cos(A-B) + a\cos(B-C) + b\cos(C-A)] = \text{area LMN} = 4\Delta,$$

and the equality is proved.

(II.) $a\cos(B-C) + b\cos(A-C) + c\cos(A-B)$

$$= 2R[\sin(B+C)\cos(B-C) + \sin(A+C)\cos(A-C) + \sin(A+B)\cos(A-B)]$$

$$= 2R\Sigma\sin 2A = 8R\sin A\sin B\sin C$$

$$= 2ab\sin C/R = 4\Delta/R.$$

16219. (S. NARAYANA AIYAR.)—In the reduction of the differential expression $dx/\sqrt{(MN)}$ to the standard form, the values given by Cayley in case (4), $MN = (1+mx^2)(1+nx^2)$ where $m > n$ for x^2, k^2, and $dx/\sqrt{(MN)}$ are $1/m$. $[y^2/(1+y^2)]$, $(m-n)/n$ and $(1/\sqrt{m})$. $(dy/\sqrt{Y})$ (Art. 414, Cayley's

Elliptic Functions, Second Edition). Show that these values are incorrect and find the correct values.

Solution by MAHENDRA NATH, D.E., M.A., B.Sc.

That the values given are incorrect is easily shown. The correct values are $1/m \,.\, [y^2/(1-y^2)]$, $(m-n)/m$ and $(1/\sqrt{m}) \,.\, (dy/\sqrt{Y})$ respectively.

The simplest way of verifying this would be to put $\sqrt{m} \,.\, x = \tan\phi$; then

$$\sqrt{m} \,.\, dx = \sec^2\phi \, d\phi, \quad 1 + mx^2 = \sec^2\phi,$$

$$1 + nx^2 = 1 + mx^2 - (m-n)\, x^2 = \sec^2\phi - [(m-n)/m] \tan^2\phi$$

$$= \{1 - [(m-n)/m] \sin^2\phi\} \sec^2\phi;$$

therefore $dx/\sqrt{(MN)} = 1/\sqrt{m} \,.\, d\phi/\sqrt{[1-(m-n)/m \sin^2\phi]}$;

therefore, if $y = \sin\phi$, we have $mx^2 = \sin^2\phi/\cos^2\phi = y^2/(1-y^2)$, and $d\phi/\sqrt{\{1-[(m-n)/m] \sin^2\phi\}}$ reduces to $dy/\sqrt{Y}$.

16252. (JAMES BLAIKIE, M.A.)—If a, b, c are the sides of a triangle ABC; p, q, r the distances of a point D from A, B, C, prove that A, B, C, D are coplanar when the following relation holds good:—

$$\Sigma\,[(a^2p^2 + b^2c^2)(q^2 + r^2)] = \Sigma\,[a^2p^2(a^2 + p^2)] + \Sigma\, a^2q^2r^2 + a^2b^2c^2.$$

Show, both geometrically and algebraically, that this condition is identical with

$$\Sigma\,[(a^2p^2 + q^2r^2)(b^2 + c^2)] = \Sigma\,[a^2p^2(a^2 + p^2)] + \Sigma\, a^2q^2r^2 + a^2b^2c^2.$$

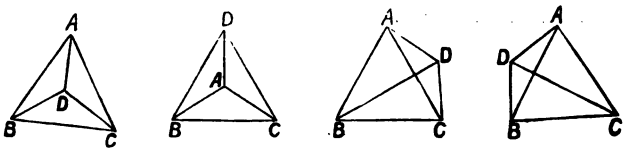
Solution by the PROPOSER.

If D is coplanar with A, B, C, either $\angle BDC + \angle CDA + \angle ADB = 2\pi$, or one of these three angles is equal to the sum of the other two. These four conditions are equivalent to the relation

$$\cos BDC = \cos(CDA \pm ADB) = \cos CDA \cos ADB \pm \sin CDA \sin ADB$$

or $(\cos BDC - \cos CDA \cos ADB)^2 = (1 - \cos^2 CDA)(1 - \cos^2 ADB)$, which reduces to

$$\cos^2 BDC + \cos^2 CDA + \cos^2 ADB = 1 + 2 \cos BDC \cos CDA \cos ADB.$$



Substituting the values of the cosines in terms of the sides, and clearing of fractions, we get

$$(q^2 + r^2 - a^2)(r^2 + p^2 - b^2)(p^2 + q^2 - c^2) + 4p^2q^2r^2$$

$$= p^2\,(q^2 + r^2 - a^2)^2 + q^2\,(r^2 + p^2 - b^2)^2 + r^2\,(p^2 + q^2 - c^2)^2,$$

which reduces to either of the conditions given above.

It is clear geometrically that the given condition is equivalent to

$$\cos \mathrm{BAC} = \cos(\mathrm{BAD} \pm \mathrm{DAC}),$$

and if we form the cosines we have a result similar to the above except that b is interchanged with q, and c with r. The algebraical expressions are proved equal by multiplying out.

16249. (W. J. Greenstreet, M.A.)—OQ is the perpendicular from the origin O to the tangent to a curve at P. If C is the centre of curvature at P, and D the foot of the perpendicular from C to OP, E, M the feet of the perpendiculars from D, O to CP, then shall QM and OE intersect in the centre of curvature at Q.

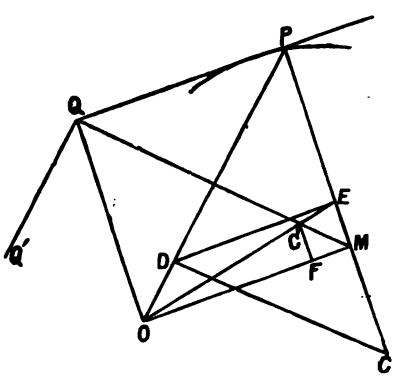
Solution by Mahendra Nath, D.E., M.A., B.Sc.

If OP, OQ $= r$, p respectively, then OQ′ the perpendicular from O on the tangent at Q $= p^2/r$; therefore $\sin \mathrm{OQQ'} = \mathrm{OQ'}/\mathrm{OQ}$ $= (p^2/r)/p = p/r = \sin \mathrm{OPQ}$. Hence since

$$\angle \mathrm{OQM} = \angle \mathrm{OPM};$$

therefore QM is the direction of the normal at Q. Now the radius of curvature at Q

$= r^2/(2r - p\,dr/dp)$
$= r^2/(2r - p/r.r\,dr/dp)$
$= \mathrm{OP}^2/(2\mathrm{OP} - \mathrm{DP})$
$= \mathrm{OP}^2/(\mathrm{OP} + \mathrm{OD})$.



But $\mathrm{QC'}/\mathrm{QM} = \mathrm{OF}/\mathrm{OM} = \mathrm{C'F}/\mathrm{EM}$ (if C′F be parallel to QO) $= \mathrm{QO}/(\mathrm{QO} + \mathrm{EM}) = \mathrm{OP}/(\mathrm{OP} + \mathrm{OD})$; therefore $\mathrm{QC'} = \mathrm{QM}.\mathrm{OP}/(\mathrm{OP} + \mathrm{OD})$ $= \mathrm{OP}^2/(\mathrm{OP} + \mathrm{OD})$ = the radius of curvature at Q; therefore C′ is the centre of curvature for Q.

7507. (R. Knowles, B.A., L.C.P.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{2\cos^2\alpha - \sin^2 2\alpha \cos^2 x}{1 - \sin^2 2\alpha \cos^2 x} \log(\cos x)\,dx = \pi \log(\tfrac{1}{2}\sec\alpha).$$

Solution by $\theta.\beta.$

The following is an alternative method of solving:—

Obviously $\dfrac{2\cos^2\alpha - \sin^2 2\alpha \cos^2 x}{1 - \sin^2 2\alpha \cos^2 x} = \dfrac{2(1 - t^2\cos 2x)}{1 + t^4 - 2t^2\cos 2x}$, where $t = \tan\alpha$.

If $|t| < 1$, we can expand in powers of t and get

$$2(1 + t^2 \cos 2x + t^4 \cos 4x + t^6 \cos 6x + \ldots);$$

while, if $|t| > 1$, we find $2(\cos 2x/t^2 + \cos 4x/t^4 + \cos 6x/t^6 + \ldots)$.
Now

$$\int_0^{\frac{1}{2}\pi} \log(4\cos^2 x)\,dx = 0 - \pi \log 2, \quad \int_0^{\frac{1}{2}\pi} \log(4\cos^2 x)\cos 2nx\,dx = (-1)^{n-1}\pi/2n,$$

and both of the above series converge uniformly with respect to x; so that we may integrate, and then we find

$$\int_0^{\frac{1}{2}\pi} \frac{2\cos^2 a - \sin^2 2a \cos^2 x}{1 - \sin^2 2a \cos^2 x} \log(\cos^2 x)\,dx = -2\pi \log 2 + \pi(t^2 - \tfrac{1}{2}t^4 + \ldots)$$

or

$$\pi[1/t^2 - \tfrac{1}{2}(1/t^4) + \tfrac{1}{3}(1/t^6) - \ldots].$$

These reduce respectively to $\pi \log \frac{1}{4}(1 + t^2) = \pi \log(\frac{1}{4}\sec^2 a)$ if $\tan^2 a < 1$ and $\pi \log(1 + 1/t^2) = \pi \log(\operatorname{cosec}^2 a)$ if $\tan^2 a > 1$.

16262. (W. P. Workman.)—If the fraction $\frac{1}{7}$ be converted into a radix fraction in scales 2, 3, 4, 5, 6, ..., the numbers of figures in the period are respectively 3, 6, 3, 6, 2, 1, 1, 3, 6, 3, 6, 2, 1, 1, ..., the numbers forming a repeating group of 7 terms. Prove this and show that the same law holds for a fraction 1/N, whether N be prime or not, if, in the latter case, the number of figures completely expressing the radix fraction be taken.

Solution by Lt.-Col. Allan Cunningham, R.E.

Let n = number of figures in period of 1/N when expressed as a radix fraction in scale of r. And let ξ = Haupt-exponent of r (mod N), *i.e.*, the least exponent giving $r^\xi \equiv +1 \pmod{N}$. Then it is known that $n = \xi$.

Now, taking all the scales $r = 2, 3, 4, \ldots, (N-1)$, it is clear that certain definite values of $n = \xi$ will occur in that range $r = 2$ to $(N-1)$, and that from $r = (N+2)$ to $(2N-1)$ these values recur, since

$$(r + N)^\xi \equiv r^\xi = +1 \pmod{N},$$

and then recur again; so that there is a period of N terms.

When N = 7, the radix fractions and periods are as follows:—

$r = 2$	3	4	5	6	7	8
$N = \cdot\dot{0}0\dot{1}$	$\cdot\dot{0}10212\dot{}$	$\cdot\dot{0}2\dot{1}$	$\cdot\dot{0}32412\dot{}$	$\cdot\dot{0}\dot{5}$	$\cdot 1$	$\cdot\dot{1}$
$n = \xi = 3$	6	3	6	2	0	1

$r = 9$	10	11	12	13	14
$N = 1\dot{2}\dot{5}$	$\cdot 142857\dot{}$	$\cdot 1\dot{6}\dot{2}$	$\cdot 186\dot{3}\dot{5}$	$\cdot 1\dot{e}$	$\cdot 2$
$n = \xi = 3$	6	3	6	2	0

It will be seen that the values of $n = \xi$ here given for the cases of $r = 7, 14$ are $n = 0, 0$ (the Proposer's figures being here 1, 1); because there is no recurrence when $r = 7, 14$.

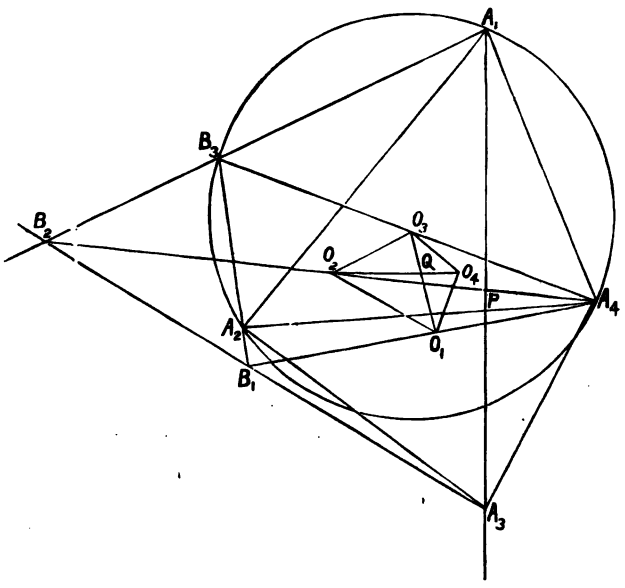
16090. (H. BATEMAN, B.A.)—A_1, A_2, A_3, A_4 are any four points in a plane. Out of these any point such as A_4 is chosen, and lines are drawn through A_1, A_2, A_3 perpendicular to A_4A_1, A_4A_2, and A_4A_3 respectively, so as to form a new triangle $B_1B_2B_3$. Prove that the ratio of the areas of the triangles $A_1A_2A_3$, $B_1B_2B_3$ is the same whichever point A_4 is chosen.

Geometrical Solutions, (I.) *by* C. HILLYER, M.A.; (II.) *by* Rev. S. SIRCOM; *Trigonometrical Solution by* Professor SANJÁNA, M.A.; *Solution by Analytical Geometry and Determinants, by* S. T. SHOVELTON, M.A., *and* R. F. WHITEHEAD, B.A.

(I.) Let O_1 be the circum-centre of the triangle $A_2A_3A_4$, O_2 of $A_3A_4A_1$, ...; then O_1O_3 is the perpendicular bisector of A_3A_4, O_2O_3 of A_4A_1, ... Let O_1O_3 and O_2O_4 meet in Q, and A_1A_3, A_2A_4 in P; then, by similar triangles, $O_2Q : QO_3 = A_1P : PA_4$ and $QO_3 : O_4Q = PA_2 : A_1P$; therefore $O_2Q : QO_4 = A_2P : PA_4$; therefore

$$\triangle O_1O_2O_3 : \triangle O_3O_4O_1 = \triangle A_1A_2A_3 : \triangle A_3A_4A_1.$$

Thus the O triangles are proportional to the corresponding A triangles; but O_1 is the mid-point of A_4B_1, O_2 of A_4B_3, and O_3 of A_4B_2; therefore $\triangle B_1B_2B_3$ is four times the triangle $O_1O_2O_3$: and so on; therefore the B triangles are also proportional to the corresponding A triangles.



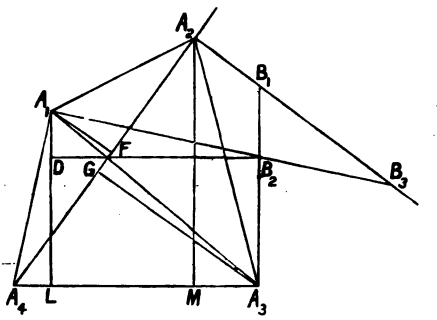
Also, by analysis, if $A_1A_2A_3$ is taken as triangle of reference and α', β', γ' the trilinear co-ordinates of the point A_4, the proof is not

difficult (though rather long) that the ratio of any one of the B triangles to the corresponding A triangle is equal to the symmetrical expression $(2R^2/\Delta^2)\,[(\beta'\gamma'\sin A_1+\gamma'\alpha'\sin A_2+\alpha'\beta'\sin A_3)^2/\alpha'\beta'\gamma']$ where Δ is the area and R the circum-radius of the triangle of reference.

(II.) $$\frac{A_1D}{LA_3}=\frac{A_4L}{A_1L}\,;\qquad \frac{A_1L-A_3B_2}{A_4A_3-A_4L}=\frac{A_4L}{A_1L}\,;$$

$$A_1L^2+A_4L^2=A_1A_4^2=A_1L.A_3B_2+A_4L.A_4A_3\,;$$

$$A_2A_4^2=A_2M.A_3B_1+A_4M.A_4A_3\,;$$



$$B_1B_3=\frac{A_1L.A_2A_4^2-A_2M.A_1A_4^2-A_4M.A_4A_3.A_1L+A_4A_3.A_4L.A_2M}{A_1L.A_2M}\,;$$

$$A_3A_4.B_1B_2=\frac{\begin{array}{r}A_3A_4.A_1L.A_2A_4^2-A_3A_4.A_2M.A_1A_4^2\\-A_2A_4^2\,(A_4M.A_1L-A_4L.A_2M)\end{array}}{A_1L.A_2M}\,;$$

$$A_4B_1B_2=\frac{A_4A_1A_3.A_2A_4^2-A_4A_2A_3.A_1A_4^2-A_4A_1A_2.A_3A_4^2}{A_1L.A_2M}$$

$$=A_4A_1A_2.A_3A_4^2\,\frac{A_4A_1A_3.A_2A_4^2-A_4A_2A_3.A_1A_4^2-A_4A_1A_2.A_3A_4^2}{4A_4A_1A_2.A_4A_2A_3.A_4A_1A_3}\,;$$

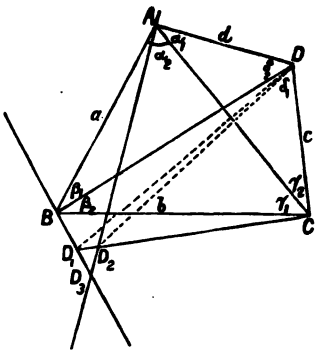
$$A_4B_2B_3=A_4A_2A_3.A_1A_4^2\,\frac{A_4A_1A_3.A_2A_4^2-A_4A_2A_3.A_1A_4^2-A_4A_1A_2.A_3A_4^2}{4A_4A_1A_2.A_4A_2A_3.A_4A_1A_3}\,,$$

... ...

Trigonometrical Solution.—The following process employs only elementary trigonometry:—

Let the quadrilateral proposed be ABCD; a_1, a_2 parts of the angle A made by AC, and so for other angles, AB = a, ... Then

$$\frac{\triangle D_1D_2D_3}{\triangle ABC} = \frac{D_1D_2 . D_2D_3 \sin D}{ab \sin B}.$$



Now

$$\begin{aligned} D_1D_2 &= CD_1 - CD_2 \\ &= c\cot\beta_2 - c\cot a_1 \\ &= c\sin(a_1-\beta_2)/\sin a_1 \sin\beta_2; \\ D_2D_3 &= d\cot\beta_1 - d\cot\gamma_2 \\ &= d\sin(\gamma_2-\beta_1)/\sin\beta_1\sin\gamma_2. \end{aligned}$$

Thus the ratio $= \dfrac{cd \sin D \sin(a_1-\beta_2)\sin(\gamma_2-\beta_1)}{ab \sin B \sin a_1 \sin\beta_1 \sin\beta_2 \sin\gamma_2}$;

but $$\frac{cd \sin D}{ab \sin B} = \frac{\triangle ADC}{\triangle ABC} = \frac{d \sin a_1}{a \sin a_2} = \frac{\sin\beta_1 \sin a_1}{\sin\delta_2 \sin a_2}.$$

Hence, finally, $$\frac{\triangle D_1D_2D_3}{\triangle ABC} = \frac{\sin(a_1-\beta_2)\sin(\gamma_2-\beta_1)}{\sin a_2 \sin\beta_2 \sin\gamma_2 \sin\delta_2}.$$

If, now, A is chosen, and lines are drawn perpendicular to AB, AC, AD to form $A_1A_2A_3$, we may show that

$$\frac{\triangle A_1A_2A_3}{\triangle BCD} = \frac{\sin(\gamma_2-\beta_1)\sin(\gamma_1-\delta_2)}{\sin a_2 \sin\beta_2 \sin\gamma_2 \sin\delta_2};$$

and this expression is the same as the former, because $\gamma_1 + \beta_2 = a_1 + \delta_2$. Hence the result.

Solution by Analytical Geometry and Determinants.—Let the coordinates of A_1, A_2, A_3, A_4 be (x_1, y_1), (x_2, y_2), ...

Then the area of $A_1A_2A_3$ is $\frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}$.

The area of $B_1B_2B_3$ is (Salmon, *Conic Sections*, § 39)

$$\begin{aligned} &\tfrac{1}{2}\begin{vmatrix} x_1-x_4, & y_1-y_4, & (x_1-x_4)^2+(y_1-y_4)^2 \\ x_2-x_4, & y_2-y_4, & (x_2-x_4)^2+(y_2-y_4)^2 \\ x_3-x_4, & y_3-y_4, & (x_3-x_4)^2+(y_3-y_4)^2 \end{vmatrix}^2 \\ &\qquad \div \begin{vmatrix} x_1-x_4, & y_1-y_4 \\ x_2-x_4, & y_2-y_4 \end{vmatrix} \cdot \begin{vmatrix} x_2-x_4, & y_2-y_4 \\ x_3-x_4, & y_3-y_4 \end{vmatrix} \cdot \begin{vmatrix} x_3-x_4, & y_3-y_4 \\ x_1-x_4, & y_1-y_4 \end{vmatrix} \\ &= \tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & x_1^2+y_1^2, & 1 \\ x_2, & y_2, & x_2^2+y_2^2, & 1 \\ x_3, & y_3, & x_3^2+y_3^2, & 1 \\ x_4, & y_4, & x_4^2+y_4^2, & 1 \end{vmatrix}^2 \div \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} \cdot \begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} \cdot \begin{vmatrix} x_3, & y_3, & 1 \\ x_1, & y_1, & 1 \\ x_4, & y_4, & 1 \end{vmatrix}. \end{aligned}$$

Therefore the ratio of the areas of $A_1A_2A_3$ and $B_1B_2B_3$ is constant.

16241. (SARADAKANTA GANGULI, M.A.) — If the Pythagorean definition of numbers in musical progression be extended to any numbers, positive or negative, integral or fractional, rational or irrational, find a series of numbers such that, if any four consecutive numbers of the series be taken, they will be in musical progression. Show that there are only two such principal series, and that all other series in musical progression are deducible from them.

Solution by R. J. WHITAKER, B.A., *and the* PROPOSER.

Let $a_1, a_2, a_3, a_4, a_5, a_6, \ldots$ be a series formed so that a_1, a_3, a_2, a_4 are in musical progression, and of the succeeding terms,

$$a_5 = a_3a_4/a_2, \quad a_6 = a_5a_4/a_3, \quad \ldots.$$

Then $$a_2 = 2a_1a_4/(a_1+a_4), \quad a_3 = \tfrac{1}{2}(a_1+a_4),$$

so that $$a_1/a_2 = a_3/a_4 \text{ and } a_5 = (a_1+a_4)^2/4a_1.$$

If a_2, a_3, a_4, a_5 are in musical progression, then must

$$a_5 + a_2 = 2a_4 \text{ and } a_3 = 2a_2a_5/(a_2+a_5).$$

The second of these conditions is equivalent to the first, since $a_5 = a_4a_3/a_2$; therefore we must have

$$(a_1+a_4)^2/4a_1 + 2a_1a_4/(a_1+a_4) = 2a_4 \quad \text{or} \quad (a_1+a_4)^3 = 8a_1a_4^2,$$

i.e., $$a_4^3 - 5a_4^2a_1 + 3a_4a_1^2 + a_1^3 = 0,$$

whence $a_4/a_1 = 1$ or $2 \pm \sqrt{5}$. If one of these conditions is satisfied, then

$$a_3 + a_6 = a_3 + a_4a_5/a_3 = (a_3^2 + a_4a_5)/a_3.$$

But $a_1a_4 = a_2a_3$ and $a_2a_5 = a_3a_4$; therefore $a_3^2 = a_1a_5$, so that

$$a_3 + a_6 = (a_1a_5 + a_4a_5)/a_3 = 2a_5,$$

since $a_1 + a_4 = 2a_3$; therefore a_3, a_4, a_5, a_6 are in musical progression, and so on for every succeeding term. Taking $a_4/a_1 = 2 \pm \sqrt{5}$, and $a_1 = 2$, the series becomes

$$2, \ 1 \pm \sqrt{5}, \ 3 \pm \sqrt{5}, \ 4 \pm 2\sqrt{5}, \ 7 \pm 3\sqrt{5}, \ 11 \pm 5\sqrt{5}, \ \ldots,$$

a recurring series of which the scale of relation is $1 - x - x^2$, and every series satisfying the conditions must have its terms proportional to one of these.

Note.—The series $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ are each geometrical progressions, with the same common ratio, namely $\frac{1}{2}(3 \pm \sqrt{5})$.

16158. (V. RAMASWAMI AIYAR, M.A.)—ABC is a triangle inscribed in a conic (focus S). If the base BC passes through the reflection of A with respect to the S directrix, then the pedal circle of the triangle ABC with respect to S has its diameter equal to SA.

Additional Solution by M. T. NARANIENGAR, M.A.

Let BC meet the S directrix at O, then, since BC passes through the reflection of A, OA and OBC are equally inclined to the directrix.

Hence, if the circle ABC cuts the conic in the fourth point D, OAD is a straight line. Now, from the harmonic property of the conic, we know the focal line perpendicular to OS is the polar of O with respect to the circle ABCD, and O and S are inverse points with respect to the circle ABCD. Thus, if X be the centre of the circle ABCD, SXBC are concyclic, and therefore, since the pedal circle of X with respect to ABC has its diameter equal to XA, the pedal circle of S has also its diameter equal to SA.

Note.—The condition that the pedal circle of S with respect to a triangle ABC should have its diameter equal to SA is at once seen to be that $\angle BSC$ equals twice the angle A. Hence $\angle BSC = \angle BXC$, and therefore B, S, X, C are concyclic.

16077. (R. W. D. Christie.)—It is known that

$$(10^{32} + 1) = 26 \pmod{97}.$$

It is required by the substitution of units for ciphers to find a square residue 49 to the same modulus.

Note by the Proposer.

It is known that $10^{32} + 1 = 26 \pmod{97}$. It is required by the substitution of units for ciphers to find a given square remainder 36 to the same modulus. In this simple case, since $10 = 10\,(3^0)$, place *one unit* in tens' place.

1. Let it be required to secure $N_1 = 1 \pmod{97}$.

We have per Question $N = 26 \pmod{97}$.

Add sum of powers of three $= 39 = 3^1 + 3^2 + 3^3$
$=$ three units in units' place.

Again add 130 $= 130 = 10\,(3^0 + 3^1 + 3^2)$
$=$ three units in tens' place.

We thus have $N = 10^{32} + 1 = 1000 \ldots 0000001 = 26 \pmod{97}$ per Quest.
Add and place units to represent 39 10101.. $= 39 \pmod{97}$.
Add and place units (one higher step) 1.1.1. $= 130 \pmod{97}$.

Now $N_1 = 10000000000000001111111 = 1 \pmod{97}$.

2. To secure $N_2 = 49 \pmod{97}$.

We have per Question
$N = 26 \pmod{97}$.

We want $23 \pmod{97} = 120$ in powers of three
$= 10\,(3^1 + 3^2) =$ placing correctly two units.

The result is $1000 \ldots 000001 = 26 \pmod{97}$,
$10100. = 23 \pmod{97}$,

i.e., $1000 \ldots 101001 = 49 \pmod{97}$.

3. To secure $N_3 = 11^2 \pmod{97}$.

We have $N = 26 \pmod{97}$.

We want $95 \pmod{97} = 192 = 289$.

Now, since 289 is suited to powers of three, we have

$$N = 10^{22} + 1 = 1000 \ldots 00000001 = 26 \pmod{97},$$

$$\dot{1}0000 = 09.$$

Here $280 = 10\,(3^0 + 3^3)$, $\dot{1}000\dot{0}1\dot{0}. = 86 = 280$,

thus $N_3 = 10000000000000010010011 = 11^2 \pmod{97}$.

In this Question it was not intended that the total number of digits should be increased or diminished, but merely interchanged. Theoretically the remainder can be got for any odd number prime or composite modulus whatever the N may be; and practically, in some cases, it is much shorter and quicker than the ordinary method of division.

16230. (Professor COCHEZ.) — On considère toutes les coniques passant par quatre points donnés. Démontrer que les polaires d'un point fixe par rapport à toutes ces coniques passent au même point.

Additional Solution by MAHENDRA NATH, D.E., M.A., B.Sc.

Let S, S′ be two fixed conics passing through the four points. Then the equation of any conic passing through the four points is $S + kS' = 0$; and, if P, P′ be the polars of the point x', y' with respect to the two fixed conics, the polar of the point x', y' with respect to the conic $S + kS' = 0$ must be the line $P + kP' = 0$, which is evidently a straight line that for all different values of k passes through a fixed point, viz., the intersection of the two fixed lines P, P′.

16261. (ROBT. W. D. CHRISTIE.)—Solve

$$x^2/z^2 - y^2/z^2 = y^2/z^2 - w^2/z^2 = 65,$$

and give an example where there are two sets of values such as

$$\left(\frac{85}{6}\right)^2 - \left(\frac{41}{6}\right)^2 = \left(\frac{113}{6}\right)^2 - \left(\frac{85}{6}\right)^2 = \left(\frac{373}{30}\right)^2 - \left(\frac{23}{30}\right)^2 = \left(\frac{527}{30}\right)^2 - \left(\frac{373}{30}\right)^2.$$

Can we substitute n = any integer for 65?

[This Question is suggested by one which I am informed appeared recently in the *Tribune*, and was answered without proof there.]

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Instead of 65, write mn: the equations to be solved are

$$x^2 - y^2 = y^2 - w^2 = mnz^2 \quad \ldots\ldots(1).$$

Hence x, y, w must satisfy $x^2 - 2y^2 = -w^2$ $\ldots\ldots(2)$.

The general solution of this Diophantine is

$$x = t^2 \mp 4tu + 2u^2, \quad y = t^2 \mp 2tu + 2u^2, \quad w = t^2 - 2u^2 \quad \ldots\ldots(3).$$

Substituting from (3) into (1), and dividing by 4 and factorising the result, it reduces to $tu\,(t+u)(t+2u) = \frac{1}{4}mns^2$.....(4),

Any integer values of tu substituted herein will satisfy equation (1); the only restriction on mn is that it must equal the product of the non-square factor contained in the product $tu\,(t+u)(t+2u)$.

Taking $mn = 65$, a few trials show that $t = 5$, $u = 4$ gives

$$tu\,(t+u)(t+2u) = 5.4.9.13 = 65.\tfrac{1}{4}.6^2.$$

These give $x = 137$, $y = 97$, $w = 7$, $s = 12$; which will be found to satisfy (1), when $mn = 65$.

A duplex solution of equation (1) with some value of mn requires

$$tu\,(t+u)(t+2u) = \tfrac{1}{4}mns^2, \text{ and } t'u'\,(t'+u')(t'+2u') = \tfrac{1}{4}mns'^2 \ldots(5).$$

A general solution of this seems difficult; but any known solution will suffice to yield an infinite series of solutions. Thus the Proposer's example yields the values of (t, u), (t', u'), ... shown below:

t, u	x, y, w	mn, s	t', u'	x', y', w'	mn, s'
7, 2	113, 85, 41	154, 6	11, 7	527, 373, 23	154, 30

This suggests writing the ratio of the two equations (5) in the form

$$\frac{t.u\,(t+2u)}{t'.u'.2} \quad \frac{t+u}{\frac{1}{2}(t'+u')(t'+2u')} = \frac{mn}{mn}\,\frac{(\frac{1}{2}s)^2}{(\frac{1}{2}s')^2} = \frac{mn}{mn}\,\frac{Z^2}{Z'^2} \quad \ldots\ldots(6),$$

writing, for shortness, $\frac{1}{2}s = Z$, $\frac{1}{2}s' = Z'$; and assume now the (trial) relations

$$tu\,(t+2u) = 2t'u' = mn \quad \ldots\ldots(7a),$$

$$t+u = Z^2, \quad \tfrac{1}{2}(t'+u')(t'+2u') = Z'^2 \quad \ldots\ldots(7b).$$

Assume further (if possible)

$$u = 2, \quad u' = t, \quad t' = t+2u = t+4 \quad \ldots\ldots(8).$$

These values (8) satisfy equation (7a), and give further

$$\tfrac{1}{2}(t'+u') = t+u = t+2, \quad t'+2u' = 3t+2u = 3t+4 \quad \ldots\ldots(8a),$$

whereby equations (7b) become

$$t+2 = Z^2, \quad 3t+4 = (Z'/Z)^2 = \zeta^2, \text{ suppose}\ldots\ldots(8b),$$

whence

$$t = Z^2 - 2 = \tfrac{1}{3}\zeta^2 - \tfrac{4}{3} \quad \ldots\ldots(8c),$$

and

$$3Z^2 - \zeta^2 = 2 \quad \ldots\ldots(9).$$

Any values of Z, ζ satisfying (9) will yield a suitable set of values of (t, u, Z) and (t', u', Z') [wherein $u = 2$ a constant, see equation (8)], satisfying equation (6); and, by equation (3), two sets (x, y, w), (x', y', w') may now be calculated satisfying equation (1) in two ways, *i.e.*, with same mn. The successive solutions of equation (9) are found to be $(Z, \zeta) = (1, 1), (3, 5), (11, 19), (41, 71), \ldots$.

The values (1, 1) do not yield a duplex solution. The next pair $Z = 3$, $\zeta = 5$ give $Z' = 15$, where [by (8), (8a–c)] $t = 7$, $u = 2$; $t' = 11$, $u' = 7$; these give the Proposer's example, as above stated. The next pair give the following results:—

Z, ζ, s'	t, u	x, y, w
11, 19, 209	119, 2	15121, 14645, 14153

t', u'	x, y, w	mn
123, 119	101999, 72725, −13193	29274

The resulting numbers $(x, y, w, mn, \ldots)$ rise so rapidly that it does not seem worth while computing any more. It is, of course, quite possible that lower solutions of equation (5) exist; but some independent mode of solution would have to be used.

The PROPOSER's discussion of the above and of a more general theorem is as follows:—

I. Let $x = 2n^2 + 4mn + m^2$, $y = 2n^2 + 2mn + m^2$, $w = 2n^2 - m^2$.

Then $$x^2 - y^2 = y^2 - w^2 = 4mn(m+n)(2n+m) = pz^2 \ldots \ldots\ldots\ldots(1).$$

Now, let $m+n = a^2$, $2n+m = p$, $mn = b^2$. Then (1) gives

$$x^2 - y^2 = y^2 - w^2 = p(2ab)^2.$$

Hence $$\left(\frac{x}{2ab}\right)^2 - \left(\frac{y}{2ab}\right)^2 = \left(\frac{y}{2ab}\right)^2 - \left(\frac{w}{2ab}\right)^2 = p.$$

Ex. gr., let $m = 9$, $n = 16$, then $a = 5$, $p = 41$, $b = 12$, $x = 1169$, $y = 881$, $w = 431$, $z = 120$. Thus

$$\left(\frac{1169}{120}\right)^2 - \left(\frac{881}{120}\right)^2 = \left(\frac{881}{120}\right)^2 - \left(\frac{431}{120}\right)^2 = 41.$$

II. Let $m+n = a^2$, $2n+m = b^2$, $mn = pz^2$. Now, if $m = 400$, $n = 41$, we get, by interchanging m, n in I.,

$$\left(\frac{387281}{24360}\right)^2 - \left(\frac{354481}{24360}\right)^2 = \left(\frac{354481}{24360}\right)^2 - \left(\frac{318319}{24360}\right) = 41.$$

III. A third method is available by assuming

$$m+n = p, \quad 2m+n = a^2, \quad mn = b^2.$$

Note (i.).—A more general theorem than I. is

$$\{q_{2n-1}n^2 - 2mn + q_{2n-3}m^2\}^2 - \{2p_{2n-2}mn - m^2 - n^2\}^2$$
$$= \{p_{2n-1}n^2 - 2mn - p_{2n-3}m^2\}^2 - \{q_{2n-1}n^2 - 2mn + q_{2n-3}m^2\}^2,$$

where $$p_n^2 - 2q_n^2 = 1.$$

(ii.) We can secure p = prime of the series $p_n = 1, 7, 41, 239, \ldots$, where $p_{n+1} = 6p_n - p_{n-1}$.

(iii.) Other endless chains can be got from $p_n = 1, 5, 29, \ldots$, of which I append an interesting example:

$$\left(\frac{48039601}{180180}\right)^2 - \left(\frac{48029801}{180180}\right)^2 = \left(\frac{48029801}{180180}\right)^2 - \left(\frac{48019999}{180180}\right)^2 = 29.$$

(iv.) In the example in Question take m, $n = 9, 5$.

(v.) A simpler solution where x and y are perfectly arbitrary appears to be $(2x^4 - 4x^2y^2 + y^4)^2 - (2x^4 - 2x^2y^2 + y^4)^2 = (2x^4 - 2x^2y^2 + y^4)^2 - (y^4 - 2x^4)^2$.

16257. (G. H. HARDY, M.A.)—Show that the series

$$\log 1 - \log 2 + \log 3 - \ldots$$

is summable, *i.e.*, that, if s_n denotes the sum of the first n terms, then $(s_1 + s_2 + \ldots + s_n)/n$ tends to a limit as n tends to infinity: and find its sum. Show also that, if $\phi(n)$ tends steadily to 0 as n tends to

infinity, the series $R_1 - R_2 + R_3 - \ldots$ where $R_n = \phi(1) + \phi(2) + \ldots + \phi(n)$ is summable, and its sum is equal to that of the convergent series

$$\phi(1) - \phi(2) + \phi(3) - \ldots .$$

Solution, with remarks, by V. RAMASWAMI AIYAR, M.A.

There is a slight inaccuracy in the statement which will be rectified. Apart from this, Mr. Hardy's conditions as to ϕ are narrower than they need be. Let $\phi_1 - \phi_2 + \phi_3 - \ldots$ be *any* convergent series, its sum being s. Let s_n denote the sum of its first n terms; so that

$$\lim_{n=\infty} s_n = s.$$

Let t_{2n-1} mean $s_1 + s_3 + \ldots + s_{2n-1}$; and t_{2n} mean $s_2 + s_4 + \ldots + s_{2n}$. Observing that t_{2n-1} and t_{2n} consist each of n terms tending to the limit s when $n = \infty$, we infer that

$$\lim_{n=\infty} t_{2n-1}/n = s \quad \text{and} \quad \lim_{n=\infty} t_{2n}/n = s.$$

Hence, whether n be odd or even, we have

$$\lim_{n=\infty} t_n/n = \tfrac{1}{2}s \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1).$$

Now let $R_n = \phi_1 + \phi_2 + \ldots + \phi_n$, $S_n = R_1 - R_2 + R_3 - \ldots \pm R_n$,

$$T_n = S_1 + S_2 + \ldots + S_n.$$

We shall show that the series $R_1 - R_2 + R_3 - \ldots$ is "summable" in the sense of Frobenius and Césaro given by Mr. Hardy, and that the sum $= \frac{1}{2}S$. This will be proved if we show that

$$\lim_{n=\infty} T_n/n = \tfrac{1}{2}s.$$

Now, the series $\phi_1 - \phi_2 + \phi_3 - \ldots$ being convergent, the power series $u \equiv \phi_1 - \phi_2 x + \phi_3 x^2 - \ldots$ is absolutely convergent ($|x| < 1$). Hence, in succession,

$$\left.\begin{aligned} u(1+x)^{-1} &= R_1 - R_2 x + R_3 x^2 - \ldots \\ u(1+x)^{-1}(1-x)^{-1} &= S_1 + S_2 x + S_3 x^2 + \ldots \\ u(1+x)^{-1}(1-x)^{-2} &= T_1 + T_2 x + T_3 x^2 + \ldots \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots (2).$$

Now the left-hand side of the above can be written $u(1-x)^{-1}(1-x^2)^{-1}$; and we have, in succession,

$$\left.\begin{aligned} u(1-x)^{-1} &= s_1 + s_2 x + s_3 x^2 + \ldots \\ u(1-x)^{-1}(1-x^2)^{-1} &= t_1 + t_2 x + t_3 x^2 + \ldots \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots (3).$$

Now, comparing (2) with (3), we get $T_n = t_n$. Hence

$$\lim_{n=\infty} T_n/n = \lim_{n=\infty} t_n/n = \tfrac{1}{2}s, \text{ by (1).}$$

In the above, let $\phi_1 = \log 1$, $\phi_2 = \log \frac{2}{1}$, $\phi_3 = \log \frac{3}{2}$, $\ldots$. Then $\phi_2, \phi_3, \phi_4, \ldots$ form a decreasing series of positive terms tending to vanish when continued indefinitely. Hence $\phi_1 - \phi_2 + \phi_3 - \phi_4 + \ldots$ is convergent. The series $R_1 - R_2 + R_3 - \ldots$ becomes

$$\log 1 - \log 2 + \log 3 \cdot \ldots .$$

Hence this series is summable, and the sum

$$= \tfrac{1}{2}[\phi_1 - \phi_2 + \phi_3 - \phi_4 + \ldots] = \tfrac{1}{2}[\log 1 - \log \tfrac{2}{1} + \log \tfrac{3}{2} - \log \tfrac{4}{3} + \ldots]$$
$$= \tfrac{1}{2}\log[\tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{4} \ldots] = \tfrac{1}{2}\log(2/\pi),$$

by Wallis's theorem.

16244. R. F. DAVIS, M.A.)—If x, y are positive quantities, prove that each of them is less than $(x^2+xy+y^2)^2/[4xy(x+y)]$.

Solution by "SOLIDUS," *and many others.*

To prove that, if x and y are positive,

$$[(x^2+xy+y^2)^2]/[4xy(x+y)] > x$$

is the same as proving that $(x^2+xy+y^2)^2-4x^2y(x+y) > 0$. But this expression may be written $(x^2-xy-y^2)^2$, which is necessarily positive.

Mr. HUGH MACCOLL, B.A., puts the Solution into the following interesting and less usual form :—

As in my *Symbolic Logic and its Applications* (Longmans), let the symbol A^P (as a *statement*) assert that A is *positive*. Then, putting F for the fraction in question, we have to prove $(F-x)^P(F-y)^P$. As x and y are interchangeable, $(F-x)^P$ implies $(F-y)^P$, so that we have only to prove $(F-x)^P$. Now, since x and y are positive,

$$(F-x)^P = [4xy(x+y)(F-x)]^P = (x^4-2x^3y-x^2y^2+2xy^3+y^4)^P$$
$$= [(x^2-xy-y^2)^2]^P,$$

which is a certainty, whether x^2-xy-y^2 be positive or negative.

16255. (ALFRED A. ROBB.)—The solution of the differential equation $d^2y/dx^2 = Ay^mx^n$ may be reduced to quadratures when either

$$n = -(m+3) \quad \text{or} \quad n = -\tfrac{1}{2}(m+3).$$

Show that the equation may also be solved when

$$m = 2,\ n = -\tfrac{15}{7};\quad m = 2,\ n = -\tfrac{20}{7} \ldots\ldots\ldots\ldots (1;\ 2).$$

Solution by the PROPOSER.

The solution in case (1) is

$$x^{\frac{1}{2}} = \int \frac{d\,(y/x^{\frac{3}{2}} + 6/49Ax^{\frac{3}{2}})}{\sqrt{[C_1 + \tfrac{28}{3}A\,(y/x^{\frac{3}{2}} + 6/49Ax^{\frac{3}{2}})^3]}} + C_2.$$

The solution in case (2) is

$$1/x^{\frac{1}{2}} = \int \frac{d\,(y/x^{\frac{3}{2}} + 6x^{\frac{3}{2}}/49A)}{\sqrt{[C_1 + \tfrac{28}{3}A\,(y/x^{\frac{3}{2}} + 6x^{\frac{3}{2}}/49A)^3]}} + C_2.$$

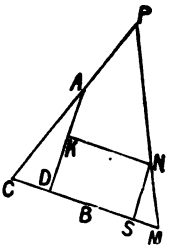
These may be verified by differentiation.

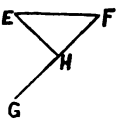
16258. (C. M. VIJENDRA RAO.)—To draw a straight line cutting any two sides of a given triangle such that the intercept of the former between the latter may be equal to each of the intercepts of the latter between the base and the required straight line.

Solutions (I.) *by* THEODORE W. HAY; (II.) *by* W. F. BEARD, M.A.; (III.) *by* MAHENDRA NATH, D.E., M.A., B.Sc.; (IV.) *by* CONSTANCE I. MARKS, B.A., *and* I. ARNOLD.

(I.) The above is a particular instance of the case in which the

intercepts of the sides and transversal are amongst themselves in the ratio of three given straight lines l, m, n.





Take a straight line AC and make angle ACB equal to one of the base angles of the given triangle. Draw AD perpendicular to BC. Take a straight line EF such that $EF : AC = n : l$, and make angle EFG equal to the remaining base angle of the given triangle. Draw EH perpendicular to FG.

In AD take DK = EH. Draw KN perpendicular to AD. Find N in KN, such that $AC : AN = l : m$. Draw NM to make with CB produced $\angle CMN = \angle EFG$.

Draw NS perpendicular to CM. Then NS = DK = EH and $\angle M = \angle F$. Therefore MN = EF.

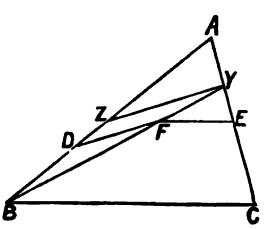
Therefore AC, AN, NM are to one another as l, m, n. Produce CA, MN to meet in P. The triangle CPM is similar to the given triangle.

Hence, to solve the problem, divide the sides of the given triangle in the ratio AC : CP and MN : NP, and the line joining the points of division is the required transversal.

When $l = m = n$, then AC = AN = NM, and the problem as given is solved.

Note.—If the given triangle has an obtuse angle at the base, then the angle ACB should be equal to the supplement of the obtuse angle, and the rest of the proof follows as above. If AN < AD ~ DK whether AC, NM be taken in order in PC, PM respectively, or in PM, PC respectively the problem is impossible for the triangle CPM with CM as base. It may, however, not be impossible with either or both of the other sides taken in succession as base.

(II.) Let ABC be a triangle with AB > AC. From AB cut off AD = AC; from CA cut off CE = BD. Draw EF parallel to BC. With centre D radius DB describe a circle cutting EF at F. Join BF and produce it to meet AC at Y. Draw YZ parallel to FD. YZ is the required line. Since FE is parallel to BC,



CY : CE = BY : BF = BZ : BD,

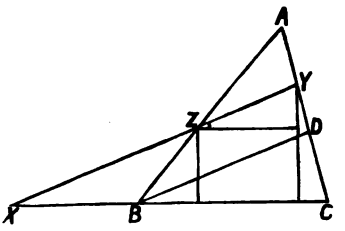
because YZ is parallel to DF; but CE = BD; therefore CY = BZ. Again, because ZY is parallel to DF, and BD = DF, therefore BZ = ZY. Thus BZ = ZY = YC.

(III.) If the required straight line makes an angle = θ with the base, and if each of the intercepts = x,

$x(\sin B + \sin\theta) = x \sin C$;

therefore

$$\sin\theta = \sin C - \sin B.$$



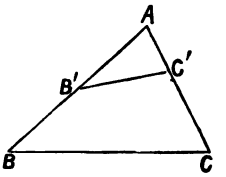
Therefore, if we draw through B the straight line BD making with BC an angle whose sine is sin C − sin B, the required straight line XYZ must be parallel to BD; therefore AZ/AY = AB/AD; but we have BX/CX . CY/AY . AZ/BZ = 1, *i.e.*,

BX/CX = AY/AZ [because CY = BZ] = AD/AB = a constant

[for BD is drawn in a fixed direction], *i.e.*, the required straight line divides CB externally in a known ratio. Whence the position of the point X is determined.

(IV.) Let ABC be the given triangle and take BC (say) as base. Then, if

$$B'B = B'C' = C'C = l,$$



we have, with the usual notation for the triangle

$$l^2 = (b-l)^2 + (c-l)^2 - 2(b-l)(c-l)\cos A;$$

therefore

$$0 = b^2 + c^2 + l^2(1 - 2\cos A) - 2bl(1-\cos A) - 2cl(1-\cos A) - 2bc\cos A,$$

or

$$0 = l^2(1 - 2\cos A) - 2l(b+c)(1-\cos A) + b^2 + c^2 - 2bc\cos A,$$

a perfectly general equation for l, since any two sides and the included angle are known in the given triangle, and the form is suitable for *any* triangle whatever. If $b = c$ and A = 60°, (i.) gives the well known value of half a side of the triangle for l. Similarly the formula may be verified for other special values of the parts of *any* given triangle.

16265. (James Blaikie, M.A.)—ABCD is a tetrahedron in which AB.CD = AC.BD = AD.BC. Prove that the 12 bisectors of the interior angles of the triangular faces intersect in 6 points on the edges of the tetrahedron, and that the straight lines joining the points of intersection lying on opposite edges are concurrent; also that the 12 bisectors of the exterior angles meet in 6 points on the edges produced, and that these 6 points are in one plane.

Solution by M. T. NARANIENGAR, M.A.

It is readily seen that the first part of the Question is true. Now, as for the second part, consider four particles at the vertices proportional to the square roots of the products of the edges of the opposite faces. Then the centre of gravity of the particles is the point of concourse of the straight lines joining opposite points.

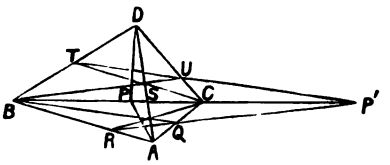
The PROPOSER and Mr. W. F. BEARD, M.A., solve the Question as follows :—

Let the internal and external bisectors of BAC meet BC in P and P'. Then, by Euclid,

$$BA/AC = BP/PC = BP'/CP'.$$

But by given property

$$BA/AC = BD/DC;$$



therefore the bisectors of BDC pass through P and P'. Similarly for the other bisectors. Also if BQ, CR be the bisectors of ABC, ACB we know that QR passes through P' (Menelaus' and Ceva's theorems). Similarly, if CT, BU be the bisectors of BCD, CBD, TU passes through P' ; therefore QR, TU are coplanar, and QT meets RU. Similarly PS meets QT and RU. Again, if P', Q', R', S', T', U' are the points where the external bisectors meet the edges, we know that P'Q'R' are in one straight line. Similarly from the triangles ACD, BCD, we know that Q'S'U', P'T'U' are in one straight line. Thus all the points P', Q', R', S', T', U' are in the plane defined by the two intersecting lines P'Q'R', Q'S'U'.

16199. (W. F. BEARD, M.A.)—Find the envelope of a straight line on which two given circles intercept equal chords.

Note by W. J. GREENSTREET, M.A.

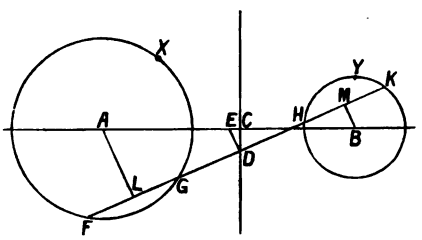
Question 8159, *Reprint*, Vol. XLV., p. 65 (by Professor Wolstenholme), ran as follows :—" Two circles with centres A and B, have radii a and b, and the distance between their centres equal to c, and $a > b$; a straight line moves so that the segments cut from it by two circles have a given ratio $\lambda : 1$; prove that (1) the envelope will be an ellipse whose major axis lies along AB, if $\lambda < 1$ or $> a/b$; and an hyperbola whose transverse axis is along AB, if $\lambda > 1$ and $< a/b$; and that if S, S' be the foci of the envelope $AS' : AS = S'B : BS = \lambda + 1 : \lambda - 1$, so that S, S' divide AB harmonically, and if C be the centre of the envelope, $CA = \lambda . CS$; (2) when $\lambda = 1$, the envelope is a parabola whose focus bisects AB, whose tangent at the vertex is the radical axis of the two circles, and latus rectum is $2(a^2 - b^2)/c$; (3) when $\lambda = a/b$, the envelope reduces to a point pair, the centres of similarity: when λ is given by the equation $(a^2 - \lambda^2 b^2)(\lambda^2 - 1) = \lambda^2 c^2$, which necessitates $a > b + c$, there is only one straight line which satisfies the condition, and the envelope

reduces to this straight line $x = \lambda^2 c/(\lambda^2-1)$; (4) if we assume λ a pure imaginary $\mu\sqrt{(-1)}$, we get for the envelope an ellipse or an hyperbola whose conjugate axis lies along the line of centres, the envelope can never become a parabola or a point pair, but may reduce to a single straight line when $\mu^2c^2 = (a^2+\mu^2b^2)(\mu^2+1)$, $(c > a+b)$; and this equation gives the maximum and minimum values of μ, but they cannot be considered geometrical."

The analytical solution was given by Prof. G. B. Mathews. The case $\lambda = 1$ is set in Smith's *Conic Sections*, chap. xii., No. 40. For the envelope when the line is divided harmonically many references might be given, *Reprint*, Vol. LXXIII., p. 128, will suffice. A very simple geometrical solution for $\lambda = 1$ is given by Mr. H. M. Taylor, in Vol. IV., p. 163, of the *Messenger*.

Solutions (I.) *by the* PROPOSER, A. M. NESBITT, M.A., *and* S. T. SHOVELTON, M.A.; (II.) *by* MAHENDRA NATH, D.E., M.A., B.Sc., *and others*; (III.) *by* L. INFELD, *and* Professor SANJÁNA, M.A.; (IV.) R. J. WHITAKER, B.A.

(I.) Let X, Y be two circles, centres A, B, CD their radical axis, E the middle point of AB. Let FGHK be a straight line such that FG, HK the chords intercepted by X, Y are equal, and let FK meet CD at E. Join ED. Draw AL, BM perpendicular to FK. Because D is on the radical axis, $DG.DF = DH.DK$; therefore $DL^2-GL^2 = DM^2-HM^2$; but GL = HM (halves of equals); therefore DL = DM and AE = EB; therefore ED is parallel to AL, BM, and is therefore perpendicular to FK. Thus FK envelopes a parabola, focus E, and with CD as tangent at the vertex.



(II.) Take the line joining the centres of the circles as the axis of x, and its mid-point as the origin, and let the co-ordinates of the centres be $(-a, 0)$, $(a, 0)$, and the equation of the variable straight line be

$$x\cos\theta + y\sin\theta - p = 0.$$

Now, since the chords intercepted are equal, the difference of the squares of the perpendiculars drawn from the centres equals the difference of the squares of the radii of the circles, *i.e.*,

$$(a\cos\theta+p)^2-(a\cos\theta-p)^2 = \text{a constant} = 4ka \text{ (suppose)};$$

therefore $p\cos\theta = k$; therefore the equation of the straight line is

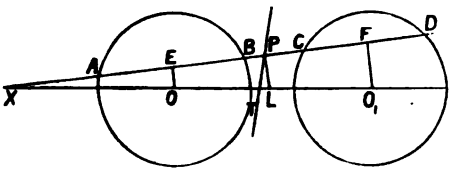
$$x\cos^2\theta + y\sin\theta\cos\theta = k \quad \text{or} \quad x\cos 2\theta + y\sin 2\theta = 2k-x;$$

therefore the envelope is

$$x^2+y^2 = (2k-x)^2 = x^2-4kx+4k^2 \quad \text{or} \quad y^2 = 4k(k-x),$$

a parabola whose relation to the two circles is easily investigated.

(III.) Two given circles OAB, OCD intersect equal chords AB, CD, on a line AD. Find envelope of AD. Produce DA to cut OO_1 produced in X. Let L be the mid-point of OO_1, and make OE, LP, O_1F



perpendicular to AD. Draw PT perpendicular to OO_1. Now $AE^2 = CF^2$. *i.e.*, $OA^2 - OE^2 = O_1C^2 - O_1F^2$, *i.e.*, $O_1F^2 - OE^2$ = constant, and from similar triangles $OE/O_1F = OX/O_1X$; therefore

$$OO_1/2LX = (O_1X - OX)/(O_1X + OX) = (O_1F - OE)/(O_1F + OE)$$
$$= (O_1F^2 - OE^2)/(O_1F + OE)^2,$$

i.e., $(O_1F + OE)^2/LX$ = constant, *i.e.*, LP^2/LX = constant, and $LP^2 = LT$, LX as LPX is a right-angled triangle; therefore LT = constant, and L is a fixed point; therefore T is a fixed point, and PT is perpendicular to XL; therefore PT is a fixed line; hence locus of foot of perpendicular from a fixed point L to line AD is a straight line TP; therefore AD envelopes a parabola whose focus is L and tangent at vertex line TP_1 radical axis (for when AD is ultimately a common tangent to the two circles, P bisects AD).

(IV.) Take the radical axis and the line of centres as axes. Then the circles have the equations

$$x^2 + y^2 + 2g_1x + c = 0, \quad x^2 + y^2 + 2g_2x + c = 0 \ldots\ldots\ldots\text{(i., ii.)}.$$

Then the perpendicular from the centre of (i.) to $x\cos\theta + y\sin\theta = p$ is of length $g_1\cos\theta + p$, and the radius of (i.) is $\sqrt{(g_1^2 - c)}$; therefore the chord cut off is $2\sqrt{[g_1^2 - c - (g_1\cos\theta + p)^2]}$. And since the chords cut off by the circles are equal

$$g_1^2 - c - (g_1\cos\theta + p)^2 = g_2^2 - c - (g_2\cos\theta + p)^2,$$

whence $p = \frac{1}{2}(g_1 + g_2)\sin^2\theta/\cos\theta$, and substituting in

$$x\cos\theta + y\sin\theta - p = 0,$$

we get $$2x\cos^2\theta + 2y\cos\theta\sin\theta - (g_1 + g_2)\sin^2\theta = 0,$$

whence the envelope is the parabola $y^2 + 2x(g_1 + g_2) = 0$.

16221. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Prove that

$$(x+1)^{x+1} - (x^2 + x + 1) \equiv 0 \bmod x^3 \text{ if } x \text{ be odd}$$

and $$\equiv 0 \bmod \tfrac{1}{2}x^3 \text{ if } x \text{ be even.}$$

Solutions (I.) *by* Professor E. B. ESCOTT *and the* PROPOSER;
(II.) *by* R. W. D. CHRISTIE.

(I.) This expression may be written

$$(x+1)^x(x+1)-(x^2+x+1)=\frac{x(x-1)}{1.2}x^2+\frac{x(x-1)(x-2)}{1.2.3}x^3+\ldots$$
$$+x^3+\frac{x(x-1)}{1.2}x^3+\ldots\equiv 0 \pmod{x^3}$$

when x is odd, or (mod $\frac{1}{2}x^3$) when x is even.

(II.) 1. To prove $(x+1)^{x+1}=x^2+x+1 \pmod{x^3}$, when x is odd. Let $x=2k+1$ say, then $(2k+2)^{2k+2}$ must $=4k^2+6k+3 \pmod{2k+1}^3$. Now giving the values $k=1, 2, 3, 4, 5, 6, \ldots$ successively, we find

$$4k^2+6k+3 = 13, 31, 57, 91, 133, 183, \ldots,$$

and the finite differences are $18, 26, 34, 42, 50, \ldots,$
$8, 8, 8, 8, \ldots,$

whence as usual $4k^2+6k+3$. That this is the correct result may be confirmed thus:—

$$(2k+2)^{2k+2}=4k^2+6k+3 \pmod{2k+1}^3,$$
$$(2k+1)(2k+2)^{2k+2}=4k^2+6k+2 \pmod{2k+1}^3,$$
$$(2k+1)^2(2k+2)^{2k+2}=(2k+1)^2 \pmod{2k+1}^3,$$

and $$(2k+1)^3(2k+2)^{2k+2}=(2k+1)^3=0 \pmod{2k+1}^3,$$

which is clearly true.

2. Let $x=$ even $=2k$ say, then

$$(x+1)^{x+1}=(2k+1)^{2k+1}=4k^2+2k+1 \pmod{4k^3}$$

by the method of increments or finite differences as above, and the result may be confirmed similarly thus

$$x(x+1)^{x+1}=8k^3+4k^2+2k=4k^2+2k \pmod{4k^3},$$
$$x^2(x+1)^{x+1}=8k^3+4k^2=4k^2 \pmod{4k^3},$$
$$x^3(x+1)^{x+1}=8k^2=0 \pmod{4k^3}.$$

3. By identical processes the following similar equations are established:—

$x=2k+1$: $(x-1)^{x-1}=(2k)^{2k}=2(k+1) \bmod (2k+1)^3$,

$x=2k$: $(2k-1)^{2k-1}=4k^3-2k-1 \pmod{8k^3}$,

$x^{x+1}=4x^3-4x^2+1 \pmod{x+1}^3$ when x is *odd*,

$x^{x+1}=x(x+2) \pmod{x+1}^3$ when x is *even*,

and the whole may be generalized if need be.

16088. (Professor NANSON.)—Find the condition that two given conics may intersect a given line in the same two points.

Solution by the PROPOSER.

The intersections of the conic

$$S=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

with the straight line $lx + my + ns = 0$ .....(2)
are given by (2) and $Bx^2 - 2Hxy + Ay^2 = 0$,
where A, B, H are the co-factors of a, b, h in the bordered discriminant of (1) and (2), provided that n is not zero. Hence, if n is not zero, the required conditions are $A/A' = H/H' = B/B'$. Similarly, if l is not zero, or if m is not zero, the conditions are

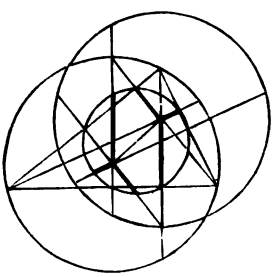
$$B/B' = F/F' = C/C' \quad \text{or} \quad C/C' = G/G' = A/A'.$$

If l, m, n are *all* different from zero, we can state the conditions in either of the equivalent symmetrical forms

$$A/A' = B/B' = C/C', \quad F/F' = G/G' = H/H'.$$

16272. (C. Joss, M.A.)—Prove that the nine-point circle passes through twelve uniformly related points.

Solution by the PROPOSER.



Describe the circum-circle and an equal circle with orthocentre as centre which may be called the converse circle. Draw perpendiculars from these centres to the sides of the triangle and produce to the circumference of the other circle both ways. The nine-point circle may easily be proved to bisect all distances from the orthocentre to the circum-circle, and from the circum-centre to the converse circle. The three additional points also form the feet of the perpendiculars on sides of a converse triangle.

9992. (Professor EMMERICH, Ph.D.)—Construct a triangle, having given the vertices A_1, B_1, C_1 of its first Brocard triangle.

Solutions (I.) *by* HENRY RIDDELL, M.E.; (II.) *by* Professor SANJÁNA, M.A.

(I.) It is only necessary to notice the fact that a first Brocard triangle is equiangular with its original, and has similarity of the so-called inverse character. We see at once that, if we construct the first Brocard triangle $A_2B_2C_2$ of $A_1B_1C_1$, we obtain a triangle directly similar to the original triangle (say ABC), and similarly placed. As this third triangle has been derived from $A_1B_1C_1$ by exactly the same construction as obtained $A_1B_1C_1$ from its equiangular original ABC,

we have $AB/A_1B_1 = A_1B_1/A_2B_2, \ldots$; remembering in naming our triangles to denote the equal angles by the same letters. A figure is unnecessary.

(II.) The angles of the triangle $A_1B_1C_1$ will be respectively those of ABC, so that A, B, C are known, and hence ω. We have

$$\sin \Omega A'C' = \sin \Omega\Omega'C' = \frac{\sin \Omega\Omega'B}{\sin \Omega B\Omega'} \sin \Omega B\Omega' = \frac{\Omega B}{\Omega\Omega'} \sin (2\omega - B)$$

$$= \frac{2R \sin C}{\sin B} \frac{\sin \omega}{2Re \sin \omega} \sin (2\omega - B) = \frac{\sin C}{\sin B} \frac{\sin (2\omega - B)}{e};$$

so also

$$\sin \Omega'C'A' = \frac{\sin A}{\sin B} \frac{\sin (2\omega - B)}{e}.$$

Thus the angles $C'A'\Omega$, $A'C'\Omega'$ being constructed from the above values, the lines $\Omega A'$, $\Omega'C'$ produced will intersect at B. Similarly it may be shown that

$$\sin \Omega B'A' = \frac{\sin A}{\sin C} \frac{\sin (C - 2\omega)}{e}, \quad \sin \Omega'A'B' = \frac{\sin B}{\sin C} \frac{\sin (C - 2\omega)}{e},$$

thus giving the lines $\Omega B'$, $\Omega'A'$ which meet at C; so also for A.

[The symbol e is put for $\sqrt{(1 - 4\sin^2\omega)}$, the eccentricity of the Brocard ellipse. Of the three angles A', B', C', two must be either less or greater than 2ω; I have taken A', C' each greater than 2ω, and B' less.]

10841. (The late Professor CLIFFORD, F.R.S.)—If, in regard to a system of n quadric surfaces, the two systems of n polar planes in regard to any two points of space are projective to one another, either the quadrics have a common Jacobian or each of them is a double plane.

Solution by R. F. WHITEHEAD, B.A.

Let $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$, $T_1 = 0$ be the polar planes of a point 1 for four of the quadrics, and let $X_2 = 0$, $Y_2 = 0$, $Z_2 = 0$, $T_2 = 0$ be the polar planes of a second point 2 for the same quadrics. Writing the equation of the polar plane of the point 1 for any fifth quadric as

$$aX_1 + bY_1 + cZ_1 + dT_1 = 0,$$

then that of the polar plane of the second point is

$$aX_2 + bY_2 + cZ_2 + dT_2 = 0.$$

This is true for *any* point 1 and for *any* point 2.

Now, if 2 be a point on the Jacobian of the first four quadrics, the planes $X_2 = 0$, $Y_2 = 0$, $Z_2 = 0$, $T_2 = 0$ have a common point, and therefore the plane $aX_2 + bY_2 + cZ_2 + dT_2 = 0$ passes through the same point. Hence, if any point lie on the Jacobian of any four quadrics of the system, then the polar planes of the point for any other four quadrics are also concurrent, that is, these other quadrics have the same Jacobian as the original four quadrics.

If the projective relation be a relation of identity, *e.g.*, when four of the quadrics are double planes, then the remaining quadrics may either belong to the system for which these planes form a self-conjugate tetrahedron, or they may be any quadrics with respect to which the polars of any two points in space are identical, *i.e.*, they may be any double planes.

10775. (J. L. MACKENZIE, B.A., B.Sc.)—Two circles cut a given circular cubic in A, B, C, D and A, B, E, F respectively; CE cuts the cubic in P, PD in Q, QF in R. Prove that DR touches the cubic at D.

[The Solver points out that in the last sentence "D" is a misprint for C.—ED.]

Solution by W. H. BLYTHE, M.A.

We note that it is proved in Salmon's *Higher Plane Curves* that CD and EF meet in some point O on the curve. Again, taking the points

O C D
E C P
F R Q

the three horizontals and two of the verticals have been given or proved collinear; therefore RCC also; therefore RC is tangent at C.

16279. (ROBT. W. D. CHRISTIE.)—Solve by an easy general process, integrally or fractionally, $X^3-pY^3=1$ where in the former case Y is > 1, as, *e.g.*, $18^3-17.7^3=1$, $\left(\frac{36}{17}\right)^3-19\left(\frac{13}{17}\right)^3=1$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

As no restriction is placed on any of X, Y, p (except that $Y>1$), we have $$X^3-1=(X-1)(X^2+X+1)=pY^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$
Hence, assuming Y any rational number, it suffices to take

$$X=1+Y^3 \quad \text{and} \quad p=X^2+X+1,$$

which obviously satisfy the equation (1).

The PROPOSER solves the equation as follows:—

For all values of a and b we have the two equations

$$\left(\frac{a^3-3ab^2-b^3}{3ab(a+b)}\right)^3-(a^3-6ab^2-3a^2b-b^3)\left(\frac{a^2+ab+b^2}{3ab(a+b)}\right)^3=1 \ldots\ldots (1),$$

$$\left(\frac{3ab(a+b)}{a^3-3ab^2-b^3}\right)^3-(b^3+6ab^2+3a^2b-a^3)\left(\frac{a^2+ab+b^2}{a^3-3ab^2-b^3}\right)^3=1 \ldots (2).$$

16090. (H. BATEMAN, B.A.)—A_1, A_2, A_3, A_4 are any four points in a plane. Out of these any point such as A_4 is chosen, and lines are drawn through A_1, A_2, A_3 perpendicular to A_4A_1, A_4A_2, and A_4A_3 respectively, so as to form a new triangle $B_1B_2B_3$. Prove that the ratio of the areas of the triangles $A_1A_2A_3$, $B_1B_2B_3$ is the same whichever point A_4 is chosen.

Additional Solution by Professor NANSON.

It is readily seen that the ratio of the areas is

$$\frac{(A_4A_1^2 \cdot A_4A_2A_3 + A_4A_2^2 \cdot A_1A_4A_3 + A_4A_3^2 \cdot A_1A_2A_4)^2}{A_4A_2A_3 \cdot A_1A_4A_3 \cdot A_1A_2A_4 \cdot A_1A_2A_3}.$$

The denominator is clearly unchanged if the suffix 4 is interchanged with any one of the suffixes 1, 2, 3. That the same is true of the numerator follows at once from the theorem of parallel axes in regard to moments of inertia.

Note.—The theorem has an obvious analogue in space of three or more dimensions. Also in the plane case it is still true if the sides of the second triangle instead of being perpendicular to, make a fixed angle in the same sense with A_4A_1, A_4A_2, A_4A_3.

Given Four Tangents to a Conic, to find the Locus of its Centre.

A new Solution by SARADAKANTA GANGULI, M.A.

Solutions of this problem are given in Dr. Salmon's and Mr. C. Smith's *Conic Sections*. But the following solution may be given which has some advantages.

Take two of the tangents for the axes of reference and let the equations of the other two be $lx+my+1=0$ and $l'x+m'y+1=0$. Let (x, y), (x_1, y_1), and (x_2, y_2) be the co-ordinates of the centre and the two foci respectively. Also let $k\sin^2\omega$ be the square of half the minor axis, where ω is the angle between the axes of reference. Then, since the product of the focal perpendiculars on any tangent is equal to the square of half the minor axis, we have

$$x_1x_2 = y_1y_2 = k \quad \ldots\ldots(1),$$

$$[(lx_1+my_1+1)(lx_2+my_2+1)]/(l^2+m^2-2lm\cos\omega) = k \quad \ldots\ldots(2),$$

and $$[(l'x_1+m'y_1+1)(l'x_2+m'y_2+1)]/(l'^2+m'^2-2l'm'\cos\omega) = k \quad \ldots(3).$$

Also, we have $$2x = x_1+x_2, \quad 2y = y_1+y_2 \quad \ldots\ldots(4).$$

From (2), we have

$$1+l(x_1+x_2)+m(y_1+y_2)+l^2x_1x_2+m^2y_1y_2+lm(x_1y_2+x_2y_1) = (l^2+m^2-2lm\cos\omega)k,$$

whence, by (4) and (1),

$$1+2lx+2my+lm(2k\cos\omega+x_1y_2+x_2y_1) = 0 \quad \ldots\ldots(5).$$

Similarly, from (3), we have

$$1+2l'x+2m'y+l'm'(2k\cos\omega+x_1y_2+x_2y_1) = 0 \quad \ldots\ldots(6).$$

Eliminating $2k\cos\omega + x_1y_2 + x_2y_1$ from (5) and (6), we have for the equation of the locus of the centre

$$l'm'(2lx + 2my + 1) = lm(2l'x + 2m'y + 1),$$

which represents a straight line passing through the point of intersection of the lines $2lx + 2my + 1 = 0$ and $2l'x + 2m'y + 1 = 0$, and this point is the middle point of that diagonal of the quadrilateral formed by the tangents, which passes through the origin. As any pair of tangents might have been taken for the axes of reference, it follows that the locus is the straight line joining the middle points of the diagonals of the quadrilateral formed by the tangents.

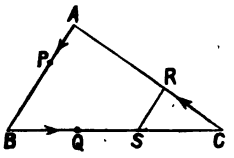
The above investigation leads to the following theorem :—The middle points of the three diagonals of a complete quadrilateral lie on a straight line.

16131. (D. Biddle.)—At the vertices of an equilateral triangle are weights of value p, q, r $(p + q = r)$, which travel along the sides in the same positive direction at a uniform pace. Trace the locus of the centre of gravity. Also extend the investigation to triangles which are not equilateral, on the understanding that, though the sides be unequal, they are traversed by the weights in equal times.

Solution by Mahendra Nath, D.E., M.A., B.Sc.

Let P, Q, R be the positions of the particles p, q, r at any instant in the more general case, and let RS, drawn parallel to AB, meet BC in S. Then, by hypothesis and the properties of similar triangles, we must have

$$\frac{AP}{c} = \frac{BQ}{a} = \frac{CR}{b} = \frac{CS}{a} = \frac{RS}{c} = \lambda \quad \text{(say)}.$$



Then, choosing BC, BA as the axes of x and y respectively, the co-ordinates of the weights p, q, r are $[0, (1-\lambda)c]$, $(a\lambda, 0)$, $[(1-\lambda)a, \lambda c]$ respectively. Therefore, if $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of gravity, we must have

$$\bar{x} = \frac{p.0 + q.a\lambda + r(1-\lambda)a}{p+q+r}; \quad \bar{y} = \frac{p(1-\lambda)c + q.0 + r.\lambda c}{p+q+r};$$

whence, eliminating λ and putting $p+q$ for r, we get

$$\frac{\bar{x}q}{a} + \frac{\bar{y}p}{c} = \frac{p^2+q^2+pq}{2(p+q)},$$

the locus is thus a straight line. This investigation obviously applies in the case of an equilateral triangle, because the sides are traversed in equal times.

16197. (L. Isserlis, B.A.)—A straight line moves so as to cut three fixed conics in six points forming an involution. Find the locus of the centre of the involution.

Solution by Professor Nanson.

Let a line through $P(xy)$ in the direction θ cut the conic $S = 0$ in Q, R. Then the condition $PQ.PR = k^2$, where k^2 is the power of the involution, gives, with the usual notation,

$$a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta = k^2S.$$

Solving for $\cos^2\theta$, $\cos\theta\sin\theta$, $\sin^2\theta$ from three equations of this type, and using the identity $\cos^2\theta\sin^2\theta = (\cos\theta\sin\theta)^2$, we get for the equation to the required locus $4(ahS)(hbS) = (abS)^2$.

The Proposer contributes the following note:—"The locus is a cubic, for the equation $4(ahS)(hbS) = (abS)^2$ is equivalent to

$$4(a, h, by^2+2gx+2fy+c)(h, b, ax^2+2gx+2fy+c) = (a, b, 2hxy+2gx+2fy+c)^2,$$

and the terms of the fourth degree are the same on both sides, viz., $4(a, b, h)^2x^2y^2$. If the conics are concentric $g = f = 0$ for each, and the locus is $4(a, h, by^2+c)(h, b, ax^2+c) = (a, b, 2hxy+c)^2$, or the conic

$$4[(ahc)(hba)x^2 + (ahb)(hbc)y^2 - (abh)(abc)xy] + 4(ahc)(hbc) - (abc)^2 = 0.$$

If the conics are coaxal as well as concentric, the locus reduces to $(abc)^2 = 0$, *i.e.*, the straight line at infinity."

11215. (Professor Lemoine.)—Sur les perpendiculaires MA', MB', MC' abaissées d'un point M sur les côtés du triangle ABC on prend respectivement les points A'', B'', C'' tels que

$$MA'.MA'' = MB'.MB'' = MC'.MC'' = \rho^2.$$

Quel que soit ρ, le triangle A''B''C'' est perspectif avec ABC et le centre de perspective est sur l'hyperbole équilatère ABCM.

Analytical Solution by W. F. Beard, M.A.

Take ABC for triangle of reference, and let co-ordinates of M be $\alpha_1\beta_1\gamma_1$. It is easily seen that the perpendiculars on AC, AB are

$$\beta_1 + \rho^2\cos C/\alpha_1, \quad \gamma_1 + \rho^2\cos B/\alpha_1.$$

Thus the equation of AA'' is

$$\beta/(\alpha_1\beta_1 + \rho^2\cos C) = \gamma/(\gamma_1\alpha_1 + \rho^2\cos B),$$

or

$$\beta(\gamma_1\alpha_1 + \rho^2\cos B) = \gamma(\alpha_1\beta_1 + \rho^2\cos C);$$

we get similar equations for BB'', CC'', and their form shows that AA'', BB'', CC'' are concurrent at the point given by

$$\alpha(\beta_1\gamma_1 + \rho^2\cos A) = \beta(\gamma_1\alpha_1 + \rho^2\cos B) = \gamma(\alpha_1\beta_1 + \rho^2\cos C) \ldots\ldots \text{(i.)}.$$

The equation of any conic through A, B, C is $\Sigma(u/a) = 0$, and if this is a rectangular hyperbola $\Sigma u \cos A = 0$. Also, if it passes through M,

$$\Sigma u\beta_1\gamma_1 = 0;$$

therefore $u/[a_1(\beta_1 \cos B - \gamma_1 \cos C)] = v/[\beta_1(\gamma_1 \cos C - a_1 \cos A)]$
$= w/[\gamma_1(a_1 \cos A - \beta_1 \cos B)]$.

Thus the equation of the rectangular hyperbola is

$$\Sigma[(\beta_1 \cos B - \gamma_1 \cos C)/a] = 0.$$

The point given by (i.) lies on this, if

$$\Sigma[a_1(\beta_1 \cos B - \gamma_1 \cos C)(\beta_1\gamma_1 + \rho^2 \cos A)] = 0,$$

if $\Sigma[a_1\beta_1\gamma_1(\beta_1 \cos B - \gamma_1 \cos C)] + \rho^2\Sigma[a_1 \cos A(\beta_1 \cos B - \gamma_1 \cos C)] = 0$.

which is obviously true. Thus the point of concurrence of AA'', BB'', CC'' lies on the rectangular hyperbola through ABCM.

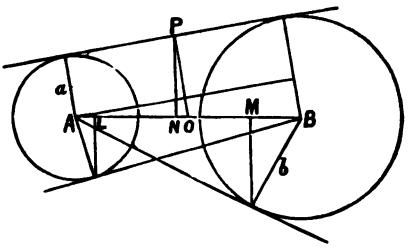
N.B.—It may be noted as a particular case of this problem that the rectangular hyperbola through A, B, C and the circum-centre must pass through the Symmedian point of the triangle.

16233. (W. F. Beard, M.A.)—The radical axis of two circles is midway between the polars of the centres of each circle with respect to the other.

[Solutions too numerous to admit of all being published.—Ed.]

Solutions (I.) *by* James Blaikie, M.A.; (II.) *by* Mahendra Nath, D.E., M.A., B.Sc.

(I.) Let a, b, c represent the radii of the circles A, B and the distance between their centres. Let L, M, N be the points where AB is met by the polars of B, A and the radical axis, O the mid-point of AB.



Then $AL = a^2/c$,

$BM = b^2/c$,

$$ON = OP \sin OPN = \tfrac{1}{2}(b+a)(b-a)/c;$$

therefore $LN = AO - AL - ON = \frac{1}{2}c - a^2/c - (b^2-a^2)/2c = (c^2-a^2-b^2)/2c$
$= \frac{1}{2}c - b^2/c + (b^2-a^2)/2c = NM.$

(II.) Taking the radical axis as the axis of y and the line joining the centres as the axis of x, the equation of the two circles will be

$x^2+y^2-2kx \pm \delta^2 = 0$ (centre k, 0), $x^2+y^2-2lx \pm \delta^2 = 0$ (centre l, 0)

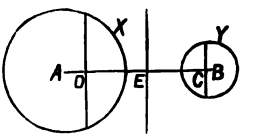
(Salmon's *Conics*, Art. 109, p. 100), where δ is the tangent from the origin to either circle. Therefore the polars are

$$(l-k)x - lk \pm \delta^2 = 0 \quad \text{and} \quad (k-l)x - lk \pm \delta^2 = 0,$$

which are evidently equidistant from the axis of y, *i.e.*, the radical axis.

The following is the PROPOSER's solution :—

Let X, Y be the two circles, A, B, their centres, r, r_1 their radii, E the foot of the radical axis, C the inverse of A with regard to Y, D the inverse of B with regard to X. Let $AB = d$,



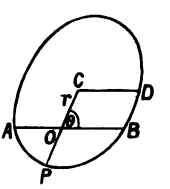
$AE^2 - EB^2 = r^2 - r_1^2$ and $AE + EB = d$; therefore $d(AE - EB) = r^2 - r_1^2$.

Thus $2d.AE = d^2 + r^2 - r_1^2$, but $AD.AB = r^2$; therefore $2d.AD = 2r^2$; therefore $2d.DE = d^2 - r^2 - r_1^2$. Similarly $2d.CE = d^2 - r^2 - r_1^2$; therefore DE = CE. That is, the polars of B to X and of A to Y are equidistant from the radical axis.

16270. (L. ISSERLIS, B.A.)—An ellipse of semi-axes a, b passes through two fixed points A, B, a distance $2d$ apart. Prove that the locus of its centre is a sextic curve, which has four nodes on the circle on AB as diameter, if $\frac{1}{2}\sqrt{(a^2+b^2)} > d > ab/\sqrt{(a^2+b^2)}$. The nodes degenerate into cusps if $d = \frac{1}{2}\sqrt{(a^2+b^2)}$, and the tangents at the cusps are parallel to AB.

Solution by M. T. NARANIENGAR, M.A.

Let r, θ be the polar co-ordinates of the centre, and ρ_1, ρ_2 the semi-conjugates CP, CD. Then obviously



(i.) $\rho_1\rho_2 \sin\theta = ab$, and (ii.) $\rho_1^2 + \rho_2^2 = a^2 + b^2$,

(iii.) $d^2/\rho_2^2 = (\rho_1^2 - r^2)/\rho_1^2$.

Thus ρ_1^2, ρ_2^2 are the roots of

(i.) $\rho^4 - (a^2+b^2)\rho^2 + (a^2b^2/\sin^2\theta) = 0$,

(ii.) $\rho^4 - \rho^2(a^2b^2/d^2\sin^2\theta) + (a^2b^2r^2/d^2\sin^2\theta) = 0$.

Eliminating ρ, we have

$$a^2b^2(r^2-d^2)^2 = [(a^2+b^2)r^2\sin^2\theta - a^2b^2][a^2b^2 - d^2\sin^2\theta(a^2+b^2)],$$

which is the sextic required. Its four nodes are such that

$$r = d = ab/[\sin\theta\sqrt{(a^2+b^2)}].$$

Hence $d > ab/\sqrt{(a^2+b^2)}$ and $< \frac{1}{2}\sqrt{(2ab)} < \frac{1}{2}\sqrt{(a^2+b^2)}$ as may be easily seen.

16223. (T. STUART, M.A., D.Sc.)—Show that there is a doubly infinite series of numbers of the form $X^5 + Y^5$ which are algebraically resoluble into two factors; and find the general forms of X and Y when this is possible. Factorise $5^5 + 17^5$, $19^5 + 7^5$, $19^5 + 23^5$, $29^5 + 25^5$.

Solution by the PROPOSER.

Since we have identically

$$X^8 + Y^8 = (X^4)^2 + (Y^4)^2 = (X^4 - Y^4)^2 + 2(X^2Y^2)^2 \quad \ldots\ldots(1)\ a,\ b,$$

the Octan is factorisable in a second quadratic form if either of the above types is obtainable. Now

$$(\theta+\phi)^8 + (\theta-\phi)^8 \equiv 2(\theta^4 + 14\theta^2\phi^2 - 63\phi^4)^2 + (16\phi^8)^2(14\theta^2 - 31\phi^2) \quad \ldots(2),$$

and hence the Octan is factorisable if

$$14\theta^2 - 31\phi^2 = \text{perfect square or} = \text{twice a perfect square.}$$

Case i.—Let $14\theta^2 - 31\phi^2 = s^2$, then, since $31 = 14.2^2 - 5^2$, the most general solution of this is given by

$$\theta = 28\alpha^2 + 2\beta^2 + 10\alpha\beta, \quad \phi = 14\alpha^2 - \beta^2, \quad s = 70\alpha^2 + 5\beta^2 + 56\alpha\beta,$$

showing that for all values of α, β the Octan

$$(42\alpha^2 + \beta^2 + 10\alpha\beta)^8 + (14\alpha^2 + 3\beta^2 + 10\alpha\beta)^8$$

is expressible in two different ways in the form $t^2 + 2u^2$, and so is algebraically factorisable.

Case ii.—Let $14\theta^2 - 31\phi^2 = 2s^2$, and it follows in the same way as in Case i. that the most general values of θ, ϕ, s are given by

$$\theta = 21\alpha^2 + 3\beta^2 + 2\alpha\beta, \quad \phi = 2(7\alpha^2 - \beta^2), \quad s = 7\alpha^2 + \beta^2 + 42\alpha\beta,$$

and the Octan $\frac{1}{2}[(35\alpha^2 + \beta^2 + 2\alpha\beta)^8 + (7\alpha^2 + 5\beta^2 + 2\alpha\beta)^8]$

is expressible in two different ways in the form $t^2 + u^2$, and so is algebraically factorisable. There are then two sets of doubly infinite series of factorisable Octans, the factors of which can be found by purely algebraic processes.

Examples of Case ii. :—

I. Let $\alpha = 1$, $\beta = -1$; then, removing the common factor 2, $\theta = 11$, $\phi = 6$, and $s = -17$. Thus $X = \theta + \phi = 17$, $Y = \theta - \phi = 5$, and

$$N = \tfrac{1}{2}(17^8 + 5^8) = 3488074033.$$

Now, from (1) a, $N = \frac{1}{2}(17^8 + 5^8) = 42073^2 + 41448^2$,

and, from (2), $N = 58752^2 + 6023^2$,

and hence $N = 769.4535857$.

II. Let $\alpha = 1$, $\beta = -3$; then, on removing the common factor 2, $\theta = 21$, $\phi = 2$, $s = -55$. Thus $X = 23$, $Y = 19$, and $N = \frac{1}{2}(19^8 + 23^8)$. From (1) a,

$$N = \tfrac{1}{2}\{279841^2 + 130321^2\} = (205081)^2 + (74760)^2,$$

and, from (2),

$$N = (21^4 + 14.42^2 - 63.2^4)^2 + (16.2^3.55)^2 \equiv (218169)^2 + (7040)^2.$$

Hence $N(1869^2 - 176^2) = (218169.1869)^2 - (205081.176)^2$,

that is, $N(5.409.1693) = 5.74332721.443852117$.

Hence $N = \frac{1}{2}(19^8 + 23^8) = 641.74332721$,

and by a well known theorem the factors of $X^8 + Y^8$ must be of the form $16i + 1$. On trial we find that 17 divides into the larger factor, and so $N = 17.641.4372513$. The complete factorisations of $7^8 + 19^8$, $29^8 + 25^8$ are easily effected in a similar manner.

16223. (T. STUART, M.A., D.Sc.)—Show that there is a doubly infinite series of numbers of the term X^8+Y^8 which are algebraically resoluble into two factors; and find the general forms of X and Y when this is possible. Factorize 5^8+17^8, 19^8+7^8, 19^8+23^8, 29^8+25^8.

Note on Factorisation of $N = x^8+y^8$ (*see Question* 16223 *above*).

By Lt.-Col. ALLAN CUNNINGHAM, R.E.

The writer introduced a highly general method before the London Mathematical Society, on January 11th, 1900, for factorization of $N = x^4+y^4$ and other quartic functions, which takes the following form when applied to 8-vans, ($N = x^8+y^8$). Write

$$N = (x^4+Cy^4)^2-y^4\{2Cx^4+(C^2-1)y^4\}$$
$$= P^2-Q^2, \text{ a difference of squares [which involves N odd],}$$

if $$2Cx^4+(C^2-1)y^4 = z^2, \text{ a perfect square} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

To find values of C for which this Diophantine is solvable, any odd 8-van number (N_0) whatever, when resolved (in any way) into a difference of squares ($P_0^2-Q_0^2$), will suffice to yield four suitable values of C when N_0 is prime (and more if N_0 is composite). Let

$$N_0 = x_0^8+y_0^8 = P_0^2-Q_0^2, \ [N_0 \text{ odd}].$$

Then $$x_0^4+Cy_0^4 = \pm P_0, \quad C = (\pm P_0-x_0^4)/y_0^4, \quad z_0 = Q_0/y_0^2 \ldots\ldots(2).$$

Here are two values of C; but as x_0, y_0 are interchangeable in N_0, two more values are given by

$$y_0^4+Cx_0^4 = \pm P_0, \quad C = (\pm P_0-y_0^4)/x_0^4, \quad z_0 = Q_0/x_0^2 \ldots\ldots(2a);$$

but the symbols x, y must not be interchanged in the general formula.

Each of these values of C leads to a solvable form of the Diophantine (1). The base-solution (x_0, y_0, z_0) being known, leads to an infinite series of solutions of each of the Diophantines by known rules: thus, writing for shortness $K = (C^2-1)$, the next solution when $2C = k^4$ is given by $kx_1 = k^4x_0^4 \sim Ky_0^4$, $y_1 = 2kx_0y_0z_0$, $z_1 = z_0^4+4Kk^4x_0^4y_0^4 \ldots(3)$. Similarly x_2, y_2, z_2 may be found; and so on, *ad inf.*

Ex.—Take $$N_0 = 1^8+2^8 = 257 = 129^2-128^2.$$

The four values of C are $+8$, $-\frac{25}{4}$, $+113$, -145. Taking $C = 8$ (given by $x_0 = 1$, $y_0 = 2$, $z_0 = 32$) leads to

$$(2x)^4+63y^4 = 3z^2, \quad \text{or} \quad x^4+63(\tfrac{1}{2}y)^4 = (\tfrac{1}{4}z)^2.$$

The next solution [see (3)] is $x_1 = 62$, $y_1 = 32$, $z_1 = 17392$. Or, cancelling 2 from x, y and 4 from z; $x_1' = 31$, $y_1' = 16$, $z_1' = 4348$. Hence

$$N_1 = 31^8+16^8; \quad P_1 = 31^4+8\cdot16^4 = 1447809, \quad Q_1 = y_1'^2z_1' = 1113088,$$

so that $N_1 = (P-Q)(P+Q) = 334721 \cdot 2560897 = 334721\,(17 \cdot 97 \cdot 1553)$. The next solution (x_2, y_2, z_2) runs into high numbers.

It will be seen that the above method applies (in detail) only to odd 8-vans (one of x, y odd, one even), because the form $N = P^2-Q^2$ involves N odd (*i.e.*, if P, Q are to be integers); but, by taking $P = \frac{1}{2}P'$, $Q = \frac{1}{2}Q'$ (where P', Q' are odd integers), it can be applied also to even 8-vans (x, y both odd). The method itself is however equally applic-

able to the resolution of N into each of the following 2-ic partitions (whether N be odd or even)

$$N = a^2 + b^2 = c^2 + 2d^2 = e^2 \sim 2f^2.$$

In each case a 4-ic Diophantine similar to (1) arises, viz.,

$$2Cx^4 + (C^2-1)\,y^4 = -s^2, \text{ or } -2s^2, \text{ or } +2s^2.$$

Suitable values of C (to render the Diophantines solvable) may be found by taking any 8-van N_0 whose (a, b), (c, d), or (e, f) is known; even the algebraic partitions of N_0, viz.,

$$N_0 = (x_0^4)^2 + (y_0^4)^2 = (x_0^4 \sim y_0^4)^2 + 2(x_0^2 y_0^2)^2 = (x_0^4 + y_0^4)^2 - 2(x_0^2 y_0^2)^2$$

will suffice. And each of them will yield four suitable values of C.

The above methods have the advantage of great generality, inasmuch as any 8-van whatever suffices to yield solvable Diophantines. They have the practical drawback of running almost at once into very high numbers.

Remarks on Dr. T. Stuart's Method of Factorisation of 8-*vans* (see Question 16223 above).

This method has the great advantage of involving only 2-ic Diophantines ($14\theta^2 - 31\phi^2 = s^2$ or $2s^2$) instead of 4-ic Diophantines (as in the writer's methods). The two 2-ic Diophantines arrived at appear however to be quite special (not general). It would be a great advantage if a method could be found of producing such 2-ic Diophantines in some general manner for the case of 8-vans.

16292. (*Communicated by* H. MacColl, B.A.)—"*Whatever P and Q may stand for, we may show, à priori, that some P is Q.* For *All PQ is Q* by the law of identity, and similarly *All PQ is P*; therefore, by a syllogism in *Darapti*, *Some P is Q.*"—How would you deal with this paradox? [From Dr. Keynes's *Formal Logic.*]

Solution by H. MacColl, B.A.

Let $\phi(P, Q)$ be any formula in logic or mathematics. If the formula holds good for all intelligible values or meanings of P and Q, it is valid; if it does not, it is invalid. For example, in mathematics the formula $P^2 - Q^2 = (P-Q)(P+Q)$ is valid; the formula $PQ = P + Q$ is not. Any argument of the form AB $\therefore$ C makes four assertions; (1) it asserts A; (2) it asserts B; (3) it asserts C; and (4), by the symbol $\therefore$ (or word "therefore") it asserts AB : C, that AB implies C. I maintain, in opposition to the generality of logicians, that *if any one of these four assertions is false, the argument is false.* On this common sense principle, the argument of the question (like all syllogistic arguments as usually given in logical text-books) is invalid. To prove this, let A denote "All PQ is Q," let B denote "All PQ is P," and let C denote "Some P is Q." The argument AB $\therefore$ C here fails when the classes

P and Q are *real but mutually exclusive;* for then A, B, C are all three false. I hold that A and B are then false because the class PQ being by hypothesis *unreal* (or non-existent) can neither be contained in the *real* class Q nor in the *real* class P; and C is false because P and Q are, by hypothesis, mutually exclusive. But, if we strike out the symbol $\therefore$ (or word "therefore") and replace it, as in my *Symbolic Logic and its Applications* (Longmans), by : (the symbol of implication), the invalid formula AB $\therefore$ C becomes then the valid formula AB : C, which only asserts that *if* all PQ is Q, and all PQ is also P, *then* some P is Q. In this valid conditional form the formula neither asserts A nor B nor C; *it only asserts that it is never the case that* (with the meanings assigned) A *and* B *are true while* C *is false*—an assertion which is incontrovertible. It thus appears that the "paradox" (like every syllogism in its customary text-book "therefore" form) is but a specious fallacy. It is incorrect to say that "whatever P and Q may stand for, we may show *a priori* that some P is Q"; but it is correct to say that whatever P and Q may stand for, *if* the class PQ is wholly contained both in the class Q and in the class P, *then* some member of the class P is also a member of the class Q. Similarly the standard syllogism "All X is Y, all Y is Z, *therefore* all X is Z," should be "*If* all X is Y, and all Y is Z, *then* all X is Z."

16248. (W. J. Dobbs, M.A.)—The edges of a tetrahedron are of lengths a, a, b, b, c, c, opposite edges being equal. Show that its volume is $\frac{1}{12}\sqrt{[2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)]}$.

[Of the many Solutions sent in it is only possible to publish a few typical ones.—Ed.]

Solutions (I.) *by* Mahendra Nath, D.E., M.A., B.Sc.; (II.) *by* Prof. Sanjána, M.A.; (III.) *by* M. T. Naraniengar, M.A., *and* A. M. Nesbitt, M.A.; (IV.) *by* W. F. Beard, M.A., *and* James Blaikie, M.A.; (V.) *by* M. S. Venkataranga Aiyar.

(I.) It is easily seen from a figure that all the plane faces of such a tetrahedron must be congruent, so that if A, B, C be the angles of the triangle that forms the base, the angles BOC, COA, AOB are equal to A, B, C respectively. Now the volume of the tetrahedron

$$= \tfrac{1}{6}abc\sqrt{(1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C)}$$

$$= \tfrac{1}{6}abc\sqrt{(4\cos A\cos B\cos C)}$$

[because, for a plane triangle,

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1]$$

$$= \tfrac{1}{12}\sqrt{(2.2bc\cos A\ 2ca\cos B\ 2ab\cos C)}$$

$$= \tfrac{1}{12}\sqrt{[2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)]}.$$

N.B.—Such a tetrahedron has the very remarkable property that the sum of the three plane angles which form each solid angle is two right angles.

(II.) The volume is given by $V^2 = -F/144$, where F is the function

$$\Sigma a^2(d^2-e^2)(d^2-f^2) + \Sigma a^2d^2(a^2-b^2-c^2) + a^2b^2c^2,$$

whose vanishing gives the relation between the six joins of four points in a plane (Salmon, §§ 52, 53). Therefore in the present case

$$V^2 = \tfrac{1}{144}[\Sigma a^4(b^2+c^2-a^2) - \Sigma a^2(a^2-b^2)(a^2-c^2) - a^2b^2c^2]$$
$$= \tfrac{1}{144}[2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)].$$

(III.) Let a sphere have its centre at a vertex of the tetrahedron; then obviously the angles of the solid angle at that vertex are together equal to two right angles. Hence, if α, β, γ denote the sides of the spherical triangle formed, $\alpha+\beta+\gamma = \pi$. Now, if θ be the altitude of the spherical triangle on the side γ,

$$\sin\theta \sin\alpha = 2n = 2\sqrt{(\cos\alpha\cos\beta\cos\gamma)} \quad (\text{since } \alpha+\beta+\gamma = \pi).$$

But, evidently, volume of the tetrahedron

$$= \tfrac{1}{3}\text{base} \times \text{alt.} = \tfrac{1}{6}abc\sin\alpha\sin\theta = \tfrac{1}{6}abc\sqrt{(\cos\alpha\cos\beta\cos\gamma)}$$
$$= \tfrac{1}{12}\sqrt{[2((a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)]},$$

since α, β, γ are equal to the angles of a plane triangle of si 'es a, b, and c.

(IV.) Let ABCD be the tetrahedron having

$$BC = AD = a, \quad CA = BD = b, \quad AB = CD = c.$$

Draw AE perpendicular to the plane BCD, and EF, EG perpendicular to BC, CD. Join AF. It is easy to prove that AF is perpendicular to BC. Thus $CF = b\cos C = 2R\sin B\cos C$ (using the usual notation for $\triangle ABC$). Similarly $CG = 2R\sin B\cos A$ and $\angle BCD = B$; therefore $EF\sin B + FC\cos B = GC$; therefore

$$EF = 2R(\cos A - \cos B\cos C) \quad \text{and} \quad AF = 2R\sin B\sin C;$$

therefore

$$AE^2 = 4R^2(\sin^2 B\sin^2 C - \cos^2 A + 2\cos A\cos B\cos C - \cos^2 B\cos^2 C)$$
$$= 4R^2(1 - \Sigma\cos^2 A + 2\Pi\cos A)$$
$$= 16R^2\Pi\cos A \quad (\text{because } \Sigma\cos^2 A = 1 - 2\Pi\cos A);$$

therefore volume of tetrahedron

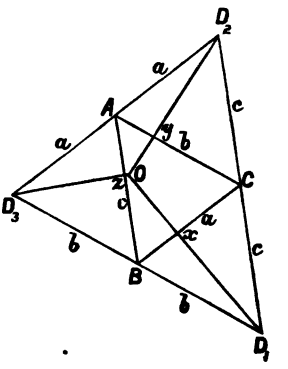
$$= \tfrac{1}{3}AE.\Delta = \tfrac{1}{3}\Delta.4R\sqrt{(\Pi\cos A)} = \tfrac{1}{6}abc\sqrt{\{[\Pi(b^2+c^2-a^2)]/8a^2b^2c^2\}}$$
$$= \tfrac{1}{12}\sqrt{[2\Pi(b^2+c^2-a^2)]}.$$

(V.) Let ABCD be the tetrahedron. Let the triangles DBC, DCA, DAB be turned about BC, CA, AB till they come to the plane of ABC, as in the figure, D_1BC, D_2CA, D_3AB representing the new positions of the triangles. Then perpendiculars D_1x, D_2y, D_3z upon BC, CA, AB will all pass through O, the foot of the perpendicular from D on the plane ABC. It is evident that

$$AD = AD_2 = AD_3,$$

and

$$p^2 = DO^2 = AD^2 - AO^2.$$



Representing the sides and angles of the triangle ABC by a, b, c and A, B, C, we have

$$Ax = a\cos B, \quad Ay = a\cos C;$$

therefore $xy^2 = a^2(\cos^2 B + \cos^2 C - 2\cos B \cos C \cos A)$;
therefore $AO^2 = 4R^2(\cos^2 B + \cos^2 C - 2\cos A \cos B \cos C)$,
where R is the radius of the circumcircle of $\triangle ABC$; therefore

$$p^2 = 4R^2(\sin^2 A - \cos^2 B - \cos^2 C + 2\cos A \cos B \cos C)$$
$$= 16R^2 \cos A \cos B \cos C.$$

Hence the volume required

$$= \tfrac{1}{3}.abc/4R.4R\sqrt{(\cos A \cos B \cos C)}$$
$$= \tfrac{1}{12}\sqrt{(2.2bc \cos A\, 2ca \cos B\, 2ab \cos C)}$$
$$= \tfrac{1}{12}\sqrt{[2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)]}.$$

Note.—The Proposer remarks that in Solution V. the work of ll. 10–21 may be shortened thus:—

$$p^2 = xD_1^2 - Ox^2 = OD_1(2xD_1 - OD_1)$$
$$= 4R \cos A.4R(\sin B \sin C - \cos A)$$
$$= 16R^2 \cos A \cos B \cos C.$$

16014. (R. F. Davis, M.A.)—Where is there to be found an investigation of the locus of the focus of a parabola which passes through three given points?

Discussion by θ. β.

The tangential equation of a parabola inscribed in the triangle of reference is $t\sqrt{\lambda} + \sqrt{\mu} - (1+t)\sqrt{\nu} = 0$,

λ, μ, ν being the perpendiculars from the angular points on the typical line. We get the focus by writing

$$\lambda = x - x_A, \quad \mu = x - x_B, \quad \nu = x - x_C,$$

where $x = \xi + i\eta$ and $\xi\eta$ are Cartesians. Take the origin at C, and the axis of ξ along CB, so that we have now $\lambda = x - a$, $\mu = x - be^{iC}$, $\nu = x$. This gives

$$\xi + i\eta = x = -\frac{(at^2 - be^{iC})^2}{4t(t+1)(at + be^{iC})} = -\frac{(at^2 - be^{iC})^2(at + be^{-iC})}{4t(t+1)(a^2t^2 + b^2 + 2abt\cos C)},$$

and so the locus required is a unicursal quintic: the real asymptotes correspond to $t = 0$, $t = \infty$, $t = -1$, which give lines parallel to the sides of the triangle. In particular, $t = \infty$ gives $\eta = \frac{1}{4}b \sin C$, so that this asymptote divides AB, AC in the ratio (3 : 1); as already proved by Mr. Youngman (see *Reprint*, N.S., Vol. XI., p. 84) for the equilateral triangle. Clearly the other asymptotes are similarly related to the other sides. However these asymptotes are *not* inflexional tangents, as in the special case, but in an *isosceles* triangle the asymptote parallel to the base is an inflexional *tangent*. The factor $a^2t^2 + b^2 + 2abt\cos C$ in the denominator indicates that the curve passes through the circular points; but the tangents there do *not* meet at the circum-centr of ABC in general. It appears th this feature is present only when ABC is equilateral.

The expression for x has two squared factors in the numerator: this shows that C is a focus, and that the lines from C to the circular points are *bitangents*. Similarly for A, B. This property agrees with Mr. Youngman's results for the equilateral triangle.

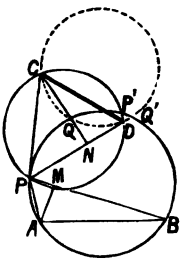
Mr. Youngman's second special case can be treated similarly to the above; but there is nothing fresh to be added, except that the origin is a cusp and (a, b) is the focus.

16110. (James Blaikie, M.A.)—Trace from geometrical considerations the locus of a point P which moves so that two given straight lines AB, CD subtend the same angle at the point, considering the cases where where ABCD is (1) a square, (2) a rectangle, (3) a parallelogram, (4) an irregular quadrilateral.

Note by C. E. Youngman, M.A.

The nature of the curves depicted in the Proposer's Solution (*Reprint*, New Series, Vol. xii., p. 44) is nicely brought out by the help of analysis.

Take the general case, and, in order that the locus may be a continuous curve, say that AB and CD must subtend equal or supplementary angles at P; then the circle PAB intersects the locus again on the circle PCD (at P′) and on the image in CD of the circle PCD (at Q and Q′). Draw AM, CN perpendicular to BP, DP. Then



$$AM : PM = CN : PN;$$

therefore

$$AM.PB : PM.PB = CN.PD : PN.PD;$$

that is, the areas PAB, PCD are in the ratio of the powers of P with respect to the circles on AB, CD as diameters. Let, then, $(\pm a, 0)$ be the co-ordinates of A and B, and $(c \pm h, d \pm k)$ those of C and D, and we have as the equation of the locus

$$ay(x^2+y^2-2cx-2dy+c^2+d^2-h^2-k^2) \pm (kx-hy-kc+hd)(x^2+y^2-a^2) = 0 \quad \ldots \quad (1),$$

the $\pm$ being necessary to account for the various signs which the areas and powers may have. Thus the locus consists in general of two circular cubics; P, P′ in the figure will lie on one of them, and Q, Q′ on the other. Also, it can be seen that AD and BC subtend equal or supplementary angles at P and P′, but not at Q, Q′; and we may infer that to any quadrangle such as ABCD there belong three cubics altogether, the complete locus of those points at which some pair or other of opposite sides of the quadrangle appear equal.

For an interesting example, let the quadrangle be orthocentric; consider its corners as the in- and ex-centres $I_1I_2I_3$ of a triangle of

reference ABC, and take Π_2 and I_1I_3 as the base pair of sides. For these, the two cubics are (in trilinears, with C and L written for $\Sigma a\beta\gamma$ and $\Sigma a\alpha$) $(C-\beta L)(\gamma+\alpha)(1+\cos B)\pm(C+\beta L)(\gamma-\alpha)(1-\cos B)=0$.

Here the upper sign gives

$$C(\gamma+\alpha\cos B)-\beta L(\gamma\cos B+\alpha)=0,$$

which simplifies into

$$-\alpha(\beta^2-\gamma^2)+\beta\cos C(\gamma^2-\alpha^2)+\gamma\cos B(\alpha^2-\beta^2)=0.$$

This form of the equation shows that the cubic is the locus of isogonal pairs in ABC whose join is perpendicular to BC; hence a proposition which it might be interesting to prove directly:—If AB, AC, touching a conic, cut a vertical tangent at B, C, and if all three touch the circles (I), (I_1), (I_2), (I_3), then Π_2 and I_1I_3 subtend equal or supplementary angles at either focus; and so do Π_3 and I_1I_2.

Returning to the original AB, CD, suppose them parallel, and accordingly put $k=0$ in equation (1). The asymptotes then turn out to be $y(h\mp a)=hd$, *i.e.*, $h(d-y)\pm ay=0$; hence they divide the distance between AB and CD in the ratio of CD to AB. (It should be remarked that the asymptotes are not generally identical with the lines IJ, KL of the Proposer's solution; his argument only proves them parallel to those lines.) Now let AB, CD, remaining parallel, become equal; this needs $h=a$. Then the asymptote of the upper sign cubic goes off to infinity, and the cubic becomes a rectangular hyperbola $d(x^2-y^2)-2cxy+y(c^2+d^2)-a^2d=0$. Once this hyperbola is suggested, it is easy enough to prove otherwise that it forms part of the locus; but to get it from the cubic is prettier, and this seems a particularly good example of the transformation.

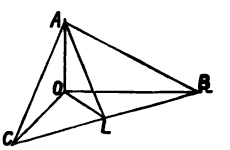
16264. (Professor NEUBERG.)—Le plan tangent en un point quelconque de la surface $xyz=k$ rencontre les axes coordonnés rectangulaires Ox, Oy, Oz aux points A, B, C. Trouver le lieu de l'orthocentre du triangle ABC.

Solutions (I.) *by* JAMES BLAIKIE, M.A.; (II.) *by* A. M. NESBITT, M.A., *and others.*

(I.) The tangent plane to the surface $xyz=k$ at the point x', y', z' is
$(x-x')y'z'+(y-y')z'x'+(z-z')x'y'=0$,
reducing to $x/x'+y/y'+z/z'=3$.
The equations to BC are therefore

$$x/x'+y/y'=3,\quad z=0;$$

and the equation to the plane through



OA perpendicular to BC is $xx'=yy'$. Thus we see that the planes $yy'=zz'$, $zz'=xx'$ pass through the orthocentre of ABC. To find the locus required we must eliminate x', y', z' from the equations

$$x/x'+y/y'+z/z'=3,\quad xx'=yy'=zz',\quad x'y'z'=k.$$

From these we obtain

$$x^2/xx' + y^2/yy' + z^2/zz' = (x^2+y^2+z^2)/xx' = 3,$$
$$xx'yy'zz' = (xx')^3 = kxyz.$$

Thus the equation of the required locus is $(x^2+y^2+z^2)^3 = 27kxyz$.

(II.) If $(\alpha\beta\gamma)$ be the point of tangency, the vertices of the triangle ABC have for co-ordinates $(3k/\beta\gamma, 0, 0)$, $(0, 3k/\gamma\alpha, 0)$, $(0, 0, 3k/\alpha\beta)$. Also the foot of the perpendicular from C on AB is the same as the foot of that from O, so that its co-ordinates at once come out to be

$$\{3k\beta/[\gamma(\alpha^2+\beta^2)],\ 3k\alpha/[\gamma(\alpha^2+\beta^2)],\ 0\}.$$

Thus the equations to the perpendicular from C on AB are

$$x/\beta = y/\alpha = (3k-\alpha\beta z)/[\gamma(\alpha^2+\beta^2)].$$

Similarly for the perpendicular from B on AC. These yield for the co-ordinates of orthocentre $3\beta\gamma k/\Sigma\beta^2\gamma^2$, $3\gamma\alpha k/\Sigma\beta^2\gamma^2$, $3\alpha\beta k/\Sigma\beta^2\gamma^2$. Thus

$$\Sigma x^2 = 9k^2/\Sigma\beta^2\gamma^2 \quad \text{and} \quad xyz = 27k^3/(\Sigma\beta^2\gamma^2)^3,$$

whence $27k^3/xyz = (9k^2/\Sigma x^2)^3$ or $27kxyz = (x^2+y^2+z^2)^3$,

the required locus.

10599. (Professor GRAM.)—Résoudre l'équation

$$[abx(x-a-b)]^{\frac{1}{3}} + [bcx(x-b-c)]^{\frac{1}{3}} + [cax(x-a-c)]^{\frac{1}{3}} = [abc(a+b+c)]^{\frac{1}{3}}.$$

Solution by CONSTANCE I. MARKS, B.A.

The given equation is

$$[abx(x-a-b)]^{\frac{1}{3}} + [bcx(x-b-c)]^{\frac{1}{3}} + [cax(x-a-c)]^{\frac{1}{3}} = [abc(a+b+c)]^{\frac{1}{3}}.$$

Assume a, b, c either *all* positive or *all* negative and the signs of the roots ambiguous. By inspection $x = a+b+c$ satisfies the equation when two of the roots on the left are taken with the same sign as that on the right and the third of opposite sign.

16301. (W. J. GREENSTREET, M.A.)—If r', ρ' are radius vector and radius of curvature of the inverse of a curve corresponding to r, ρ, find $r/\rho + r'/\rho'$ in terms of the angle between the tangent and radius vector of the curve or its inverse.

Solution by S. T. SHOVELTON, M.A.

From the equation $p/r = p'/r'$, we have

$$1/r^2(r\,dp - p\,dr) = 1/r'^2(r'dp' - p'dr'),$$

or
$$dr/r^2(r^2/\rho - p) = dr'/r'^2(r'^2/\rho' - p') \quad \ldots\ldots(1).$$

Now, since $rr' = \kappa^2$, $dr/r + dr'/r' = 0$.

Therefore, from (1), $r/\rho + r'/\rho' = p/r + p'/r' = 2\sin\phi$.

Note on the Nine-Point Circle.

By W. GALLATLY.

Let D, E, F be the mid-points of ABC ($a > b > c$); D', E', F' the feet of the perpendiculars.

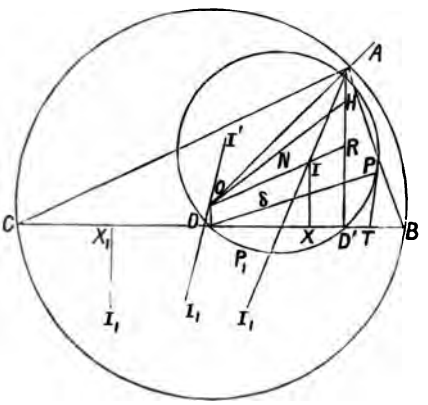
Let P, P_1, P_2, P_3 be the Feuerbach points where the in- and ex-circles touch the nine-point circle, centre N, and let the in-circle cut DP again in ∂.

Then, since P is a centre of similitude for the circles I and N,

$$D\partial : DP = R-2r : R;$$

$$\therefore\ D\partial . DP : DP^2 = R^2-2Rr : R^2; \quad \therefore\ DX/f = DP/R : (OI = f);$$

$$\therefore\ OIX \text{ or } ORD' = DPT \text{ (PT is a tangent).}$$



Also $OAR = B-C = DPD'$; $\therefore$ $PDT = D'PT = AOR$ or AOI.

Similarly $P_1DT = AOI'$; $\therefore$ $\frac{1}{2}PNP_1 = PDP_1 = IOI' = OIX - OI_1X_1$.

But
$$\tan OIX = \frac{DX}{IX-OD} = \frac{\sin B-\sin C}{\cos B+\cos C-1},$$

$$\tan OI_1X_1 = \frac{DX_1}{I_1X_1+OR} = \frac{\sin B-\sin C}{\cos B+\cos C+1};$$

$$\therefore\ \tan \tfrac{1}{2}PNP_1 = \tan PDP_1 = \tan IOI' = \frac{2(\sin B-\sin C)}{1-2\cos A}.$$

Similarly
$$\tan \tfrac{1}{2}P_2NP_3 = \tan P_2DP_3 = \frac{2(\sin B+\sin C)}{1+2\cos A}.$$

By solving the triangles INI_1 and I_2NI_3,

$$\sin PDP_1 = \sin \tfrac{1}{2}PNP_1 = \frac{\frac{1}{2}(b-c)}{\sqrt{(dd_1)}} \quad (NI = d),$$

$$\sin P_2DP_3 = \sin \tfrac{1}{2}.P_2NP_3 = \frac{\frac{1}{2}(b+c)}{\sqrt{(d_2d_3)}}.$$

Or the sines may be deduced from the tangents by employing the formula $4dd_1/R^2 = (1-2r/R)(1+2r_1/R)$

$$= (1-8\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C)(1+8\sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C)$$

$$= 4(\sin B-\sin C)^2+(1-2\cos A)^2,$$

and
$$4d_2d_3/R^2 = 4(\sin B+\sin C)^2+(1+2\cos A)^2.$$

Hence $\quad$ Area of $P_1P_2P_3 = \frac{1}{16}.\frac{(b+c)(c+a)(a+b)}{d_1d_2d_3}\,R^2$.

And, if Q be the area of the quadrilateral $PP_1P_2P_3$,

$$Q = \frac{R^2}{16}\,\frac{(b+c)(c+a)(a+b)}{d_1d_2d_3} + \frac{R^2}{16}\,\frac{(a-b)(b-c)(c+a)}{dd_1d_3}\,;$$

$\therefore\ 16.dd_1d_2d_3/[R^2(a+c)].Q$

$= d(b+c)(a+b) + d_2(a-b)(b-c)$

$= b(a+c)(R-r+r_2) - (b^2+ac)(r_2+r)$

$= b(a+c)\,R(1+4\sin^2\frac{1}{2}B) - (b^2+ac)\,4\sin\frac{1}{2}B\,(a+c)/b\sin\frac{1}{2}B$

$= R(a+c)(a-c)^2/b\,;$

$\therefore\ Q = \frac{1}{16}R^2/b\,(a^2-c^2)^2/dd_1d_2d_3$

16260. (R. F. Davis, M.A.) — If $x-M$, $x-N$, $x-(2M-N-2)$, $x-(2N-M-2)$ are all perfect squares, find x in terms of M and N.

Solution by B. C. Wallis, F.C.P., B.Sc. (Econ.).

Let $\quad M = N+2n+1,$

$x-(2N-M-2) = A^2,\ x-N = B^2,\ x-M = C^2,\ x-(2M-N-2) = D^2;$

then $\quad x = \frac{1}{2}(M-N-1),$

$A^2-B^2 = 2n+3,\ B^2-C^2 = 2n+1,\ C^2-D^2 = 2n-1;$

therefore $C^2 = n^2,\ D^2 = (n-1)^2,\ B^2 = (n+1)^2,\ A^2 = (n+2)^2;$

therefore $\quad x-M = n^2 = [\frac{1}{2}(M-N-1)]^2;$

therefore $\quad x = \frac{1}{4}(M^2+N^2+1-2MN+2N+2M).$

Whence

$$\left.\begin{aligned} x-N &= \tfrac{1}{4}(M-N+1)^2 \\ x-(2N-M-2) &= \tfrac{1}{4}(M-N+3)^2 \\ x-(2M-N-2) &= \tfrac{1}{4}(N-M+3)^2 \end{aligned}\right\}\ \text{perfect squares.}$$

16242. (Lt.-Col. Allan Cunningham, R.E.)—Factorise completely (into prime factors) $\quad N = (3^{54}+2^{54})$.

Completion of the Solution by the Proposer.

The solution previously published was *incomplete* in that the large factor M = 574952041 was left undetermined in character. It has since been found that M = 13177·43633.

It may be of interest to show how this was found out.

Euler's Rule gives that every prime factor (p) of M must be of form $p = 108w+1$, since M is a factor of $(3^{54}+2^{54})/(3^{18}+2^{18})$. There are 71 primes of this form $\not> \sqrt{M}$. About two-thirds of these may be rejected by the rules for the 2-ic, 4-ic, 8-ic, and 3-ic residuacity of 2, 3, and 6 as

follows. The symbols ω, ϵ denote *odd* and *even* numbers respectively hereafter.

Note that $3^{54} \equiv -2^{54} \pmod{p = 108\varpi + 1}$. Raising this to the power $(p-1)/108$,

$$(3/p)_3 = (1/p)_3 . (2/p)_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

But $(3/p)_3 = +1$, and $(1/p)_3 . (2/p)_3 = +1$, whether ϖ be $= \omega$ or ϵ.

Hence the 2-ic test (1) is identically satisfied, so that ϖ may $= \omega$ or ϵ.

Next, when $\varpi = \epsilon$, it will be found by similar reasoning that the following tests must be satisfied.

$(6/p)_4 = -1$, if $p = 8.27\omega + 1$; $\quad (6/p)_4 = +1$, if $p = 8.27\epsilon + 1$;

$(6/p)_3 = -(2/p)_4$, if $p = 16.27\omega + 1$; $\quad (6/p)_3 = +(2/p)_4$, if $p = 16.27\epsilon + 1$.

Further the cubic tests must be satisfied, which will be found to be

$$(12/p)_3 \neq 1, \text{ if } p = 4.27\varpi + 1, \text{ with } \varpi \neq 3k.$$

$$(12/p)_3 = 1, \text{ if } p = 4.27\varpi + 1, \text{ with } \varpi = 3k.$$

These tests will be found to reject 47 out of the 71 trial divisors, thus leaving only 27 for actual trial by division.

The smallest one ($p = 109$) may be also rejected because it divides into L (and L, M are mutually prime). The 13th trial gives $p = 13177$ a divisor.

10502. (H. W. Segar.)—(1) show that the curve

$$r^4 - 2c^2r^2 \cos 2\theta + c^4 = 4a^2c^2 \sin^2\theta$$

has two conjugate points other than the origin, and the normals drawn from them are of length $2a$; and (2) draw a figure of the curve.

Note by A. M. Nesbitt, M.A.

If we change to Cartesians we find

$$\partial u/\partial x = x\,(6x^4 + 12x^2y^2 + 6y^4 - 8c^2x^2 + 2c^4),$$

$$\partial u/\partial y = y\,(6x^4 + 12x^2y^2 + 6y^4 + 8c^2y^2 + 2c^4 - 8a^2c^2),$$

so that, if both are zero either $x = y = 0$ or $x^2 + y^2 = a^2$. But this circle meets the curve at points given by

$$2c^2y^2 = a^4 + c^4, \quad 2c^2x^2 = -(a^2 - c^2)^2,$$

so that $\partial u/\partial x$ and $\partial u/\partial y$ are simultaneously zero only at the origin—so far as concerns real points. Imaginary conjugate points seem to me a quite needless complication, and, in fact, are a contradiction in terms.

16276. (C. M. Ross.)—If

$$\begin{vmatrix} \sin 2\theta, & \cos\theta, & \sin\theta, & 1 \\ \sin 2\phi, & \cos\phi, & \sin\phi, & 1 \\ \sin 2\psi, & \cos\psi, & \sin\psi, & 1 \\ \sin 2\chi, & \cos\chi, & \sin\chi, & 1 \end{vmatrix} = 0$$

and if θ, ϕ, ψ, χ are different positive angles each less than 2π, prove that $\theta + \phi + \psi + \chi = n\pi$ (n an odd integer).

Solution by S. T. Shovelton, M.A., *and* N. Kuppuswami, B.A.

θ, ϕ, ψ, χ are roots of an equation of the form

$$a \sin 2x + b \cos x + c \sin x + d = 0,$$

which may be expressed as $as^4 + \ldots - a = 0$, where $s = e^{2ix}$, *i.e.*, $s_1 s_2 s_3 s_4 = -1$, *i.e.*, $e^{2i(\theta+\phi+\psi+\chi)} = -1$; therefore

$$\theta + \phi + \psi + \chi = (2m+1)\pi.$$

16280. (Professor A. Droz-Farny.)—Si P_λ/Q_λ est une réduite de rang λ de la fraction continue $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}$, démontrer que

$$P_\lambda = Q_{\lambda-1} + Q_\lambda, \qquad P_\lambda = P_{\lambda-1} + 2Q_{\lambda-1}.$$

Solutions (I.) *by* R. W. D. Christie; (II.) *by* N. Kuppuswami, B.A.; (III.) *by* C. M. Ross.

(I.) On a par la théorie générale $P^2_{\lambda-1} - 2Q^2_{\lambda-1} = \pm 1$. Conséquemment $(2Q_{\lambda-1} + P_{\lambda-1})^2 - 2(P_{\lambda-1} + Q_{\lambda-1})^2 = \mp 1$, car par l'addition on fait $0 = 0$. Ainsi

$$P_\lambda = 2Q_{\lambda-1} + P_{\lambda-1}, \qquad Q_\lambda = P_{\lambda-1} + Q_{\lambda-1} \ldots\ldots\ldots\ldots (1, 2)$$

de là $$P_\lambda = Q_{\lambda-1} + Q_\lambda \ldots\ldots\ldots\ldots\ldots\ldots (1, 2),$$

qui prouvent les théorèmes demandés.

(II.) $$Q_\lambda = 2Q_{\lambda-1} + Q_{\lambda-2};$$

therefore

$$Q_\lambda/Q_{\lambda-1} = 2 + Q_{\lambda-2}/Q_{\lambda-1} = 2 + \frac{1}{(Q_{\lambda-1}/Q_{\lambda-2})} = 2 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} \ldots (A)$$

for $Q_2/Q_1 = 2/1$; therefore

$$Q_{\lambda-1}/Q_\lambda = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2},$$

the number of elements being $\lambda - 1$; therefore

$$Q_{\lambda-1}/Q_\lambda + 1 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = P_\lambda/Q_\lambda;$$

therefore $$Q_{\lambda-1} + Q_\lambda = P_\lambda \ldots\ldots\ldots\ldots\ldots\ldots (B).$$

Again, from (A),

$$Q_\lambda/Q_{\lambda-1} = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = 1 + P_{\lambda-1}/Q_{\lambda-1};$$

therefore $Q_\lambda = Q_{\lambda-1} + P_{\lambda-1}$; therefore $P_\lambda = P_{\lambda-1} + 2Q_{\lambda-1}$ from (B).

(III.) Since P_λ/Q_λ is the λ-th convergent,

$$P_\lambda = 2P_{\lambda-1} + P_{\lambda-2}, \qquad Q_\lambda = 2Q_{\lambda-1} + Q_{\lambda-2} \ldots\ldots\ldots\ldots (1, 2)$$

From (1) it is clear that the numerators of the convergents form a recurring series, any three consecutive terms being connected by

$$1 - 2x - x^2 = 0.$$

If S is the sum of this series,

$$S = \frac{\alpha+1}{2(\alpha-1)}\,\frac{1}{1-\alpha x} + \frac{\alpha-3}{2(\alpha-1)}\,\frac{1}{1+x/\alpha},$$

where $\alpha - 1/\alpha = 2$. The coefficient of $x^{\lambda-1}$ is

$$\frac{\alpha+1}{2(\alpha-1)}\,\alpha^{\lambda-1} + \frac{\alpha-3}{2(\alpha-1)}\,(-1)^{\lambda-1}\,\frac{1}{\alpha^{\lambda-1}} = P_\lambda.$$

Similarly, if S′ is the sum of the recurring series formed by the denominators of the successive convergents,

$$S' = \frac{\alpha}{2(\alpha-1)}\,\frac{1}{1-\alpha x} + \frac{\alpha-2}{2(\alpha-1)}\,\frac{1}{1+x/\alpha}.$$

Hence $$\frac{\alpha}{2(\alpha-1)}\,\alpha^{\lambda-1} + \frac{\alpha-2}{2(\alpha-1)}\,(-1)^{\lambda-1}\,\frac{1}{\alpha^{\lambda-1}} = Q_\lambda.$$

Again

$$\begin{aligned} Q_{\lambda-1} + Q_\lambda &= \frac{\alpha}{2(\alpha-1)}\,\alpha^{\lambda-2}(\alpha+1) + \frac{\alpha-2}{2(\alpha-1)}\,(-1)^{\lambda-2}\,\frac{1}{\alpha^{\lambda-2}}\left(1-\frac{1}{\alpha}\right) \\ &= \frac{\alpha+1}{2(\alpha-1)}\,\alpha^{\lambda-1} + \frac{\alpha-3}{2(\alpha-1)}\,(-1)^{\lambda-1}\,\frac{1}{\alpha^{\lambda-1}} \\ &= P_\lambda. \qquad [\text{since } (\alpha-2)(1-\alpha) \equiv \alpha-3] \end{aligned}$$

Again $$P_\lambda - P_{\lambda-1} = \frac{\alpha+1}{2(\alpha-1)}\,\alpha^{\lambda-2}(\alpha-1) + \frac{\alpha-3}{2(\alpha-1)}\,\frac{(-1)^{\lambda-2}}{\alpha^{\lambda-2}}\left(1+\frac{1}{\alpha}\right) \text{ (A)}.$$

Also $$2Q_{\lambda-1} = \frac{\alpha}{\alpha-1}\,\alpha^{\lambda-2} + \frac{\alpha-2}{\alpha-1}\,(-1)^{\lambda-2}\,\frac{1}{\alpha^{\lambda-2}} \quad \ldots\ldots\ldots\ldots \text{ (B)}.$$

The difference between the first terms of the dexters of (A) and (B) is

$$\frac{\alpha+1}{2(\alpha-1)}\,\alpha^{\lambda-2}(\alpha-1) - \frac{\alpha}{\alpha-1}\,\alpha^{\lambda-2} = \frac{\alpha^{\lambda-2}}{\alpha-1}\,(\alpha^2-2\alpha-1) = 0,$$

since $\alpha - 1/\alpha = 2$. Also, the difference between the second terms is

$$\begin{aligned} &\frac{\alpha-3}{2(\alpha-1)}\,\frac{(-1)^{\lambda-1}}{\alpha^{\lambda-1}}(1+\alpha) - \frac{\alpha-2}{\alpha-1}\,(-1)^{\lambda-2}\,\frac{1}{\alpha^{\lambda-2}} \\ &\qquad = \frac{3(-1)^{\lambda-1}}{(\alpha-1)\,\alpha^{\lambda-2}}\,(\alpha^2-2\alpha-1) = 0; \end{aligned}$$

therefore $P_\lambda - P_{\lambda-1} - 2Q_{\lambda-1} = 0$; therefore $P_\lambda - P_{\lambda-1} = 2Q_{\lambda-1}$.

16265. (Professor SANJÁNA, M.A.)—Given that the roots of the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$ are $\alpha^2, \beta^2, \gamma^2, \ldots, \nu^2$, is there any simple process by which equations of the same degree with roots $\alpha, \beta, \gamma, \ldots, \nu$ and $-\alpha, -\beta, -\gamma, \ldots, -\nu$ can be written down? [*E.g.*, the roots of $64x^3 - 112x^2 + 56x - 7 = 0$ are $\sin^2\frac{2}{7}\pi$, $\sin^2\frac{4}{7}\pi$, $\sin^2\frac{8}{7}\pi$; the equations with the square roots taken positively and negatively as roots are $8x^3 - 4\sqrt{7}x^2 + \sqrt{7} = 0$, $8x^3 + 4\sqrt{7}x^2 - \sqrt{7} = 0$.]

Solution by W. F. BEARD, M.A.

It is clear that there is no such process as desired, for if we obtain equations $(x-\alpha)(x-\beta)\dots(x-\nu)=0$, $(x+\alpha)(x+\beta)\dots(x+\nu)=0$, we must also obtain other pairs of equations such as

$$(x+\alpha)(x-\beta)\dots(x-\nu)=0 \quad \text{and} \quad (x-\alpha)(x+\beta)\dots(x+\nu)=0,$$

and all such similar combinations, the number of which provides an interesting example in combinations.

11643. (R. W. D. CHRISTIE.)—Show how to arrange the signs so as to make the following equations true :—

$$1^n \pm 2^n \pm 3^n \pm \dots \pm (2^{n+1})^n = 1^n \pm 3^n \pm 5^n \pm \dots \pm (2^{n+2}-1)^n = 0.$$

Ex. gr. : $-1^2+2^2+3^2-4^2+5^2-6^2-7^2+8^2$
$$= -1^2+3^2+5^2-7^2+9^2-11^2-13^2+15^2 = 0.$$

Solutions (I.) *by* "SOLIDUS"; (II.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(I.) The two theorems in the Question are included in the more general theorem. If $\phi_n(m)$ be any rational integral function of m of the n-th degree, then

$$\sum_{m=1}^{m=2^{n+1}} [\pm\phi_n(m)] = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(i.)},$$

provided the signs are taken in the following order :—for two terms $+-$; for four terms, the additional signs are these reversed, giving $+--+$; for eight terms, the additional signs are these reversed, giving $+--+-++-$, and so on. Thus

$$\sum_{m=1}^{m=2^{n+1}} [\pm\phi_n(m)] = \sum_{m=1}^{m=2^n} [\pm\phi_n(m)] - \sum_{m=1}^{m=2^n} [\pm\phi_n(m+2^n)]$$

$$= \sum_{1}^{2^n} \{\pm[\phi_n(m)-\phi_n(m+2^n)]\}$$

$$= \sum_{1}^{2^n} [\pm\psi_{n-1}(m)] \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(ii.)},$$

which is of the same form as the sinister of (i.).

Hence, by continued application of (ii.), we have

$$\sum_{1}^{2^{n+1}} [\pm\phi_n(m)] = \sum_{1}^{2^n} [\pm\psi_{n-1}(m)] = \sum_{1}^{2^{n-1}} [\pm\chi_{n-2}(m)] = \dots = \sum_{1}^{2} [\pm f_0(m)]$$

which is obviously $= 0$.

N.B.—To get a formula for the sign attached to $\phi(m)$, let m be expressed in the binary scale, so that

$$m = a_0 + a_1 2 + a_2 2^2 + a_3 2^3 + \dots + a_r 2^r.$$

Then the sign to be prefixed to $\phi(m)$ is $(-)^{a_1+a_2+a_3+\ldots+a_r+\nu}$, when λ is zero excepting when $a_0 = 0$, and the first of the quantities $a_1, a_2, \ldots$ which does not vanish has an even suffix. In such cases λ is $= 1$.

(II.) The following is a particular solution when $n = 2$. Let ν = number of terms $= 2^3 = 8$. Let $x_1^2, x_2^2, \ldots, x_\alpha^2$ be the + squares, and let $y_1^2, y_2^2, \ldots, y_\beta^2$ be the − squares. Taking the first case, then

$$\Sigma(x^2) + \Sigma(y^2) = 1^2 + 2^2 + 3^2 + \ldots + 8^2 = \tfrac{1}{6} \cdot 8 \cdot 9 \cdot 17 = 204.$$

But $$\Sigma(x^2) - \Sigma(y^2) = 0;$$

therefore $$\Sigma(x^2) + \Sigma(y^2) = 102.$$

Also $$\Sigma(x) + \Sigma(y) = 1 + 2 + 3 + \ldots + 8 = \tfrac{1}{2} \cdot 9 \cdot 8 = 36.$$

Assume now that (if possible) there are the number (α) of the x^2 is equal to the number (β) of the y^2, and also that $\Sigma(x) = \Sigma(y)$. These give

$$\alpha = \beta = 4, \quad \Sigma(x) = \Sigma(y) = 18.$$

Hereby the Question is reduced to solving

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 = 18,$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 = 102,$$

where to $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ must be given the values 1 to 8.

A few trials suffice to give the Proposer's solution

$$2 + 3 + 5 + 8 = 1 + 4 + 6 + 7 = 18,$$
$$2^2 + 3^2 + 5^2 + 8^2 = 1^2 + 4^2 + 6^2 + 7^2 = 102.$$

The second example in odd numbers may be solved in a similar way.

The problem in cubes ($n = 3$) seems much more difficult. Here $\nu = \alpha + \beta = 16$ in the first case

$$\Sigma(x^3) = \Sigma(y^3) = \tfrac{1}{2}(1^3 + 2^3 + 3^3 + \ldots + 16^3) = 9248,$$
$$\Sigma(x) + \Sigma(y) = \tfrac{1}{2} \cdot 17 \cdot 16 = 136.$$

But it does not seem easy to assume other conditions to aid the solution.

Note by the PROPOSER.

The solution depends on the theorem

$$\Sigma({}^nS_1)^m - \Sigma({}_nS_2)^m + \Sigma({}_nS_3)^m - \ldots \pm \Sigma({}_nS_n)^m \equiv 0,$$

where $m < n$ (*v. Lond. Math. Soc. Proc.*, Vol. XX., p. 119), and the use of the values $a = 2^0$, $b = 2^1$, $c = 2^2$, $d = 2^3$, $\ldots k = 2^n$; *e.g.*, using the convention $abcd = a + b + c + d$, we have

$$(abcd)^3 - (abc)^3 - (abd)^3 - (acd)^3 - (bcd)^3 + (ab)^3 + (ac)^3 + (ad)^3 + (bc)^3 + (bd)^3 + (cd)^3 - a^3 - b^3 - c^3 - d^3 = 0,$$

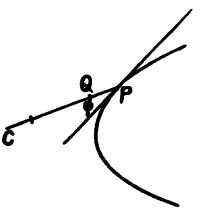
i.e., $$15^3 - 14^3 - 13^3 - 11^3 - 7^3 + 12^3 + 10^3 + 9^3 + 6^3 + 5^3 + 3^3 - 8^3 - 4^3 - 2^3 - 1^3 = 0, \ldots.$$

16267. (A. M. NESBITT, M.A.)—A variable conic, which is (*a*) an equilateral hyperbola, (*b*) a parabola, is drawn to have four-pointic

contact with the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Show that the locus of the centre of (a) is a bicircular quartic which is the inverse of the fixed hyperbola with respect to a concentric circle of radius $\sqrt{(a^2-b^2)}$; and that the locus of the focus of (b) is also a bicircular quartic, which is its inverse with respect to a concentric circle of radius $\sqrt{(a^2+b^2)}$.

Solution by M. T. NARANIENGAR, M.A.

(i.) The first part of the Question is the same as Question 26 in Edwards' *Differential Calculus*, ch. x., and may be solved thus :—We know that four-pointic contact with a given conic means that the variable conic and the given conic have their centres in a line with the point of contact. Let therefore Q be the centre of the equilateral hyperbola in question, then



$$\rho = CD^2/ab = CD^2/(CP.CD\sin\phi) = CD^2/CP\sin\phi$$

for the given hyperbola, and for the equilateral

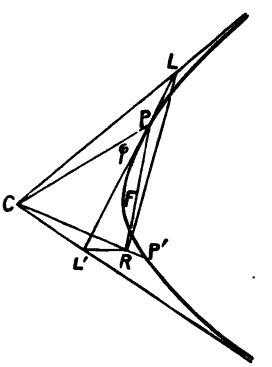
hyperbola $\rho = QP^2/QP\sin\phi = QP/\sin\phi.$

Hence $CD^2 = CP.QP;$

therefore $CP.CQ = CP^2 - CP.QP = CP^2 - CD^2 = a^2 - b^2.$

Thus the locus of Q is the inverse of that of P, viz., the given hyperbola.

(ii.) The second part of the Question is wrong. The property required to be proved is perhaps the following :—If the tangent at P meet the asymptotes at L, L', and CP' equal to CP cut the circle LCL' in R, then it is evident that CL'R and CPL are similar triangles. For CP, CP' are equally inclined to CL, CL', and



$$\angle CPL = \angle PL'C + \angle L'CP$$
$$= \angle PL'C + \angle RCL = \angle CL'R.$$

Hence $CL.RL' = CR.PL,$

and therefore

$$CL'.LR = CR.PL,$$

i.e., CL'R and LPR are also similar triangles. Thus PL' bisects

$$\angle CPR \text{ and } PL'^2 = CP.PR.$$

Now the focus of the parabola having four-pointic contact with the given hyperbola at P is easily seen to be the mid-point of PR, since CP is the diameter through P, and the focal chord of curvature is

$$2\rho\sin\phi = 2CD^2/CP = 2PL'^2/CP = 2PR = 4FP,$$

if F be the focus. Again, $CR.CP' = CL.CL' = a^2 + b^2$; therefore R is the inverse of P' with respect to a circle of radius $\sqrt{(a^2+b^2)}$, &c.

INDEX.

The numbers refer to the pages, and in the case of Questions and Solutions the reference is, in general, to the page bearing the Question.

(*a*)—SUBJECTS.

Algebra (General), 32, 40, 40, 52, 56, 71, 75, 81, 90, 104, 106, 108, 109, 110.

Algebra (Higher), 40, 45, 47, 54, 60, 66, 72, 87, 89, 101, 107.

Arithmetic and Theory of Numbers, 19, 24, 27, 34, 35, 42, 43, 52, 62, 63, 71, 76, 77, 86, 90, 95, 97, 98, 106.

Calculus, Differential and Integral, 36, 70, 89, 104, 107, 112.

Combinations, Permutations, and Allied Algebra, 32, 110.

Co-ordinate Geometry, Plane—of Point, Straight Line, Circle, and Conic (Problems connected with), 18, 21, 26, 30, 33, 41, 44, 45, 53, 67, 72, 77, 84, 87, 91, 92, 93, 93, 94, 95, 101, 102.

Co-ordinate Geometry of Surfaces whose Algebraic Equations are of the First or Second Degree, 48, 59, 66, 89, 103.

Curves, Plane (exclusive, in general, of Circle and Conic), 18, 21, 26, 33, 51, 55, 63, 70, 90, 93, 95, 101, 102, 104, 107, 111.

Differential Equations, 19, 33, 38, 81.

Elliptic Functions, 68.

Formal Logic, 98.

Functions, Theory of, 50.

Geometrical Conics, 18, 21, 30, 41, 44, 53, 58, 59, 67, 75, 84, 111.

Geometrography, 44.

Geometry, Elementary—of Straight Lines in a Plane, of Plane Rectilineal Figures, and of the Circle, 22, 36, 52, 53, 54, 58, 68, 70, 72, 81, 84, 84, 88, 88, 89, 91, 94, 99, 102, 105.

Geometry, Higher and Modern, 22, 22, 26, 41, 44, 46, 59, 59, 63, 67, 75, 83, 84, 84, 88, 89, 91, 93, 93, 94, 102, 105.

Geometry in n Dimensions ($n > 2$), 51, 58, 83, 91, 99.

Integrals, Definite (Problems in or solved by), 70.

Maxima and Minima, 53, 85.

Mechanics, 58, 84, 91, 92.

Minima and Maxima, 53, 85.

Permutations, Combinations, and Allied Algebra, 32, 110.

Probabilities, 65.

Series, 17, 38, 42, 46, 49, 56, 59, 70, 75, 79, 108, 110.

Surfaces (exclusive of those whose Algebraic Equations are of the First or Second Degree), 66, 103.

Symbolic Logic and its Applications to Mathematical Problems, 81, 98.

Theory of Numbers and Arithmetic, 19, 24, 27, 34, 35, 42, 43, 52, 62, 63, 71, 76, 77, 86, 90, 95, 97, 98, 106.

Transformations of Special Types (Problems connected with), 52, 63, 75, 94, 104, 111.

Trigonometry, Plane (Problems in or solved by), 22, 37, 67, 68, 69, 69, 72, 81, 88, 93, 93, 99, 105, 107.

Triangles, Spherical (Problems connected with), 58, 99.

(*b*)—CONTRIBUTORS.

Aiyar, M. S. Venkataranga, 99.
Aiyar, S. Narayana, 32, 36, 39, 49, 65, 68.
Aiyar, V. Ramaswami, M.A., 65, 75, 80.
Apte, R. N., 53.
Arnold, I., 81.
Arunachalan, M. V., M.A., 17, 22, 32, 42.
Bagchi, S. C., B.A., LL.B., 33.
Barniville, J. J., B.A., I.C.S., 59.
Bateman, H., B.A., 72, 91.
Beard, W. F., M.A., 21, 30, 31, 33, 41, 53, 59, 81, 84, 84, 85, 93, 94, 95, 99, 110.
Bell, A. H., 35.
Benny, L. B., 41.
Biddle, D., 36, 92.
Blaikie, J., M.A., 30, 34, 54, 69, 83, 84, 94, 99, 102, 103.
Blythe, W. H., M.A., 37, 90.
Bool, G., B.A., 52.
Bordage, Prof., 38.
Bowman, Rev. J. C., 68.
Christie, R. W. D., 35, 42, 62, 76, 77, 79, 87, 90, 90, 108, 110, 111.
Clifford, the late Prof., F.R.S., 89.
Cochez, Prof., 41, 77.
Cunningham, Lt.-Col. A., R.E., 19, 24, 25, 27, 34, 35, 42, 43, 52, 63, 71, 77, 86, 87, 90, 97, 98, 106, 110.
Davis, R. F., M.A., 18, 81, 101, 106.
Dick, Hon. G. R., M.A., 45, 46, 47, 66.
Dobbs, W. J., M.A., 18, 99, 101.
Droz-Farny, Prof. A., 67, 108.
Emmerich, Prof., Ph.D., 88.
Escott, Prof. E. B., 37, 39, 87.
Gallatly, W., M.A., 105.
Ganguli, S., M.A., 75, 91.
Genese, Prof. R. W., M.A., 58.
Gram, Prof., 104.
Greenstreet, W. J., M.A., 18, 51, 56, 70, 84, 104.
Hardy, G. H., M.A., 42, 79.
Hay, T. W., 81.
Hillyer, C. E., M.A., 72.
Hudson, Prof. W. H. H., M.A., 51.
Infeld, L., 85.
Isserlis, L., B.A., 17, 26, 93, 93, 95.
Joss, C., M.A., 88.
Julian, J., 37.
Knowles, R., B.A., 68, 70.
Kuppuswami, N., B.A., 103, 108.
Lampe, Dr. E., 46.
Lauvernay, Prof., 54, 55.
Lemoine, Prof., 67, 92.
MacColl, H., B.A., 81, 98.
Mackenzie, J. L., B.A., B.Sc., 90.
Marks, C. I., B.A., 53, 81, 104.
Menon, N. P., 22, 23.
Mukherji, L. M., M.A., B.L., 48.
Nanson, Prof., 30, 40, 47, 51, 53, 66, 87, 91, 93.
Naraniengar, M. T., M.A., 18, 41, 53, 58, 75, 84, 95, 99, 112.
Narayanan, S., B.A., L.T., 18, 38, 41, 59.
Nath, M., D.E., M.A., B.Sc., 39, 69, 70, 77, 81, 85, 92, 94, 99.
Nesbitt, A. M., M.A., 21, 30, 33, 47, 48, 54, 59, 60, 85, 99, 103, 107, 111.
Neuberg, Prof., 21, 36, 36, 37, 44, 51, 103.
Polignac, Prince C. de, 55.
Rao, C. M. Vijendra, 81.
Reeves, F. W., M.A., 18.
Riddell, H., M.E., 22, 37, 41, 56, 68, 88.
Roach, Rev. T., M.A., 46.
Robb, A. A., 81.
Roberts, W. D., 52.
Ross, C. M., 39, 44, 45, 56, 59, 107, 108.
Sanjána, Prof., M.A., 17, 21, 24, 38, 40, 41, 42, 53, 58, 59, 63, 67, 72, 85, 88, 99, 109.
Scott, R. F., M.A., 54.
Segar, H. W., 107.
Shovelton, S. T., M.A., 26, 45, 58, 72, 85, 104, 108.
Sircom, Rev. S., 19, 72.
Stuart, T., M.A., D.Sc., 19, 27, 43, 95, 96, 97.
Tavani, F., 50.
Walker, J. J., F.R.S., 19.
Wallis, B. C., F.C.P., B.Sc. (Econ.), 106.
Whitaker, R. J., B.A., 17, 22, 33, 59, 68, 75, 85.
Whitehead, R. F., B.A., 72, 89.
Workman, W. P., M.A., 71.
Youngman, C. E., M.A., 44, 64, 102.

θ. β., 70, 101.
"Solidus," 81, 110.

Printed by C. F. Hodgson & Son, 2 Newton Street, Kingsway, W.C.

Royal 8vo, 967 pp., with 285 Diagrams, price 34s.

A SYNOPSIS

OF ELEMENTARY RESULTS IN

PURE MATHEMATICS,

Containing Six Thousand Propositions, Formulæ, and methods of Analysis, with abridged Demonstrations; supplemented by an Index to the articles on Pure Mathematics which have appeared in thirty-two BRITISH AND FOREIGN JOURNALS AND TRANSACTIONS *during the past century, containing about* 14,000 *references.*

By G. S. CARR, M.A.,

Late Prizeman and Scholar of Gonville and Caius College, Cambridge.

The work may also be had in Sections separately, as follows:—

		s.	d.
Section I.	—MATHEMATICAL TABLES; including C. G. S. units. Least Factors from 1 to 99,000. The Gamma-function, &c.	2	0
„ II.	—ALGEBRA	2	6
„ III.	—THEORY OF EQUATIONS AND DETERMINANTS	2	0
„ IV.	—PLANE TRIGONOMETRY { together; with	2	0
„ V.	—SPHERICAL do. { 22 Diagrams		
„ VI.	—ELEMENTARY GEOMETRY; with 47 Diagrams	2	6
„ VII.	—GEOMETRICAL CONICS; with 35 Diagrams	2	0
„ VIII.	—DIFFERENTIAL CALCULUS	2	0
„ IX.	—INTEGRAL CALCULUS	3	6
„ X.	—CALCULUS OF VARIATIONS	3	6
„ XI.	—DIFFERENTIAL EQUATIONS		
„ XII.	—CALCULUS OF FINITE DIFFERENCES		
„ XIII.	—PLANE COORDINATE GEOMETRY, with 182 Diagrams	6	0
„ XIV.	—SOLID COORDINATE GEOMETRY, with 26 Diagrams	3	6

N.B.—The Index will not be sold separately, but possessors of previous sections may complete the volume by returning these to the publisher to be bound up with the remaining sections and the Index.

The work is in part designed for the use of University and other Candidates who may be reading for examination. It forms a digest of the contents of ordinary treatises and is arranged so as to enable the student rapidly to revise his subjects. To this end all the important propositions in each branch of Mathematics are presented within the compass of a few pages. This has been accomplished, firstly, by the omission of all extraneous matter and redundant explanations, and, secondly, by carefully compressing the demonstrations in such a manner as to place only the leading steps of each prominently before the reader. The whole is intended to form a permanent work of reference for mathematicians.

NOTICES OF THE PRESS.

"The completed work admirably serves the object Mr. Carr set before himself. Every subject that can be classed under the head of Pure Mathematics, with the exception perhaps of Quaternions, appears to us to have been carefully treated on the author's lines."—*The London, Edinburgh, and Dublin Philosophical Magazine.*

"We believe the book to be singularly free from errors. The author is to be heartily congratulated on the successful termination of his labours. The work is printed in a good bold type on good paper, and the figures are admirably drawn."—*Nature.*

"By the boldly printed reference numbers the reader can, if he pleases, go back and back upon the links in the chain of demonstration from the highest theorem to elementary principles. Repetition, otherwise necessary, is thereby avoided, and at the same time the connection of theorems and the course of evolution of formulæ are more clearly exhibited. Although the object of the synopsis is to supplement only, and not to supersede, the usual text-books, the clearness of statement is such that, without any other book, a student who has already some knowledge of mathematics will find it a great saving of time to use this volume as a guide to higher attainments. We cannot adequately set forth in this notice the high estimation we have of this work. The engravings in it are numerous and well executed, especially those referring to geometry and geometrical conics, where a distinguishing thick line is employed with excellent effect. The work is beautifully printed."—*Engineering.*

"The idea of the work is well conceived and is well carried out. In its completed form it is as full, and yet as concise, an index as could be desired of all the most important propositions in the extensive field of Pure Mathematics."—*The Academy.*

New and Revised Edition, small crown 8vo, price 2s.

AN INTRODUCTION TO GEOMETRY.

FOR THE USE OF BEGINNERS.

CONSISTING OF

EUCLID'S ELEMENTS, BOOK I.

ACCOMPANIED BY NUMEROUS EXPLANATIONS, QUESTIONS, AND EXERCISES.

BY JOHN WALMSLEY, B.A.

This work is characterized by its abundant materials suitable for the training of pupils in the performance of *original work*. These materials are so graduated and arranged as to be specially suited for class-work. They furnish a copious store of useful examples from which the teacher may readily draw more or less, according to the special needs of his class, and so as to help his own method of instruction.

OPINIONS OF THE PRESS.

"We cordially recommend this book. The plan adopted is founded upon a proper appreciation of the soundest principles of teaching. We have not space to give it in detail, but Mr. Walmsley is fully justified in saying that it provides 'for a natural and continuous training to pupils taken in classes.'"—*Athenæum.*

"The book has been carefully written, and will be cordially welcomed by all those who are interested in the best methods of teaching Geometry."—*School Guardian.*

"Mr. Walmsley has made an addition of a novel kind to the many recent works intended to simplify the teaching of the elements of Geometry. . . . The system will undoubtedly help the pupil to a thorough comprehension of his subject."—*School Board Chronicle.*

"When we consider how many teachers of Euclid teach it without intelligence, and then lay the blame on the stupidity of the pupils, we could wish that every young teacher of Euclid, however high he may have been among the Wranglers, would take the trouble to read Mr. Walmsley's book through before he begins to teach the First Book to young boys."—*Journal of Education.*

"We have used the book to the manifest pleasure and interest, as well as progress, of our own students in mathematics ever since it was published, and we have the greatest pleasure in recommending its use to other teachers. The *Questions* and *Exercises* are of incalculable value to the teacher."—*Educational Chronicle.*

KEY to the above, price 3s.

Crown 8vo, price 1s. 6d.

THE ELEMENTS OF LOGARITHMS; with the Sandhurst and Woolwich Questions for 1880-87. By W. GALLATLY, M.A.

Demy 8vo, price 5s.

ALGEBRA IDENTIFIED WITH GEOMETRY, in Five Tracts. By ALEXANDER J. ELLIS, F.R.S., F.S.A.

Demy 8vo, price 5s. each.

TRACTS relating to the MODERN HIGHER MATHEMATICS. By the Rev. W. J. WRIGHT, M.A.

TRACT No. 1.—DETERMINANTS. TRACT No. 2.—TRILINEAR COORDINATES. TRACT No. 3.—INVARIANTS.

The object of this series is to afford to the young student an easy introduction to the study of the higher branches of modern Mathematics. It is proposed to follow the above with Tracts on Theory of Surfaces, Elliptic Integrals, and Quaternions.

Demy 8vo, price 7s. 6d. Third Edition.

(*Used as the Text-book in the Royal Military Academy, Woolwich.*)

LECTURES on the ELEMENTS of APPLIED MECHANICS. Comprising—(1) Stability of Structures; (2) Strength of Materials. By MORGAN W. CROFTON, F.R.S., late Professor of Mathematics and Mechanics at the Royal Military Academy. Revised by H. HART, M.A.

Third Edition. Extra fcap. 8vo, price 4s. 6d.

ELEMENTARY MANUAL of COORDINATE GEOMETRY and CONIC SECTIONS. By Rev. J. WHITE, M.A.

MATHEMATICAL WORKS

PUBLISHED BY

FRANCIS HODGSON,

89 FARRINGDON STREET, E.C.

In 8vo, cloth, lettered.

PROCEEDINGS of the LONDON MATHEMATICAL SOCIETY.

Vol. I., from January 1865 to November 1866, price 10s.
Vol. II., from November 1866 to November 1869, price 16s.
Vol. III., from November 1869 to November 1871, price 20s.
Vol. IV., from November 1871 to November 1873, price 31s. 6d.
Vol. V., 1873–74, price 15s.
Vol. VI., 1874–75, price 21s.
Vol. VII., 1875–76, price 21s
Vol. VIII., 1876–77, price 21s.
Vol. IX., 1877–78, price 21s.
Vol. X., 1878–79, price 18s.
Vol. XI., 1879–80, price 12s. 6d.
Vol. XII., 1880–81, price 16s.
Vol. XIII., 1881–82, price 18s.
Vol. XIV., 1882–83, price 20s.
Vol. XV., 1883–84, price 20s.
Vol. XVI., 1884–85, price 20s.
Vol. XVII., 1885–86, price 25s.
Vol. XVIII., 1886–87, price 25s.
Vol. XIX., 1887–88, price 30s.
Vol. XX., 1888–89, price 25s.
Vol. XXI., 1889–90, price 25s.
Vol. XXII., 1890–91, price 27s. 6d.
Vol. XXIII., 1891–92, price 20s.
Vol. XXIV., 1892–93, price 22s.
Vol. XXV., 1893–94, price 22s.
Vol. XXVI., 1894–95, price 30s.
Vol. XXVII., 1895–96, price 35s.
Vol. XXVIII., 1896–97, price 32s. 6d.
Vol. XXIX., 1897–98, 2 parts, price 20s. each.
Vol. XXX., November 1898 to March 1899, price 20s.
Vol. XXXI., April to December 1899, price 20s
Vol. XXXII., January to June 1900, price 20s.
Vol. XXXIII., November 1900 to February 1901, price 20s.
Vol XXXIV., March 1901 to April 1902, price 20s.
Vol. XXXV., May 1902 to January 1903, price 20s.

SECOND SERIES.—Vol. I., 1903, price 25s.
Vol. II., 1904, price 25s.
Vol. III., 1905, price 25s.
Vol. IV., 1907, price 25s.
Vol. V., 1907, price 25s.

In half-yearly Volumes, demy 8vo, price 6s. 6d. each.
(To Subscribers, price 5s.)

MATHEMATICAL QUESTIONS, with their SOLUTIONS, Reprinted from the EDUCATIONAL TIMES. Edited by CONSTANCE I. MARKS, B.A.

Of this series eighty-eight volumes (Old Series, 75 Volumes; New Series, 13 Volumes) have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription price.

Crown 8vo, cloth, price 12s. net.

QUADRATIC PARTITIONS. By Lieut.-Col. ALLAN CUNNINGHAM, R.E.

Thirteenth Edition, small crown 8vo, price 2s. 6d.

AN INTRODUCTORY COURSE OF PLANE TRIGONOMETRY AND LOGARITHMS.

By JOHN WALMSLEY, B.A.

"This book is carefully done; has full extent of matter, and good store of examples."—*Athenæum.*

"This is a carefully worked out treatise, with a very large collection of well-chosen and well-arranged examples."—*Papers for the Schoolmaster.*

"This is an excellent work. The proofs of the several propositions are distinct, the explanations clear and concise, and the general plan and arrangement accurate and methodical."—*The Museum and English Journal of Education.*

"The explanations of logarithms are remarkably full and clear. . . . The several parts of the subject are, throughout the work, treated according to the most recent and approved methods. . . . It is, in fact, a book for *beginners*, and by far the simplest and most satisfactory work of the kind we have met with."—*Educational Times.*

A KEY

to the above, containing Solutions of all the Examples therein, price Five Shillings (net).

By the same Author.

New Edition, fcap. 8vo, cloth, price 5s.

PLANE TRIGONOMETRY AND LOGARITHMS.

FOR SCHOOLS AND COLLEGES. Comprising the higher branches of the subject not treated in the elementary work.

"This is an expansion of Mr. Walmsley's 'Introductory Course of Plane Trigonometry,' which has been already noticed with commendation in our columns, but so greatly extended as to justify its being regarded as a new work. . . . It was natural that teachers, who had found the elementary parts well done, should have desired a completed treatise on the same lines, and Mr. Walmsley has now put the finishing touches to his conception of how Trigonometry should be taught. There is no perfunctory work manifest in this later growth, and some of the chapters—notably those on the imaginary expression $\sqrt{-1}$, and general proofs of the fundamental formulæ—are especially good. These last deal with a portion of the recent literature connected with the proofs for $\sin(A+B)$, &c., and are supplemented by one or two generalized proofs by Mr. Walmsley himself. We need only further say that the new chapters are quite up to the level of the previous work, and not only evidence great love for the subject, but considerable power in assimilating what has been done, and in representing the results to his readers."—*Educational Times.*

IN PREPARATION.

By the same Author.

Suitable for Students preparing for the University Local and similar Examinations.

AN INTRODUCTION TO MECHANICS.

With numerous Examples, many of which are fully worked out in illustration of the text.

Seventh Edition, crown 8vo.

MODERN SIDE ARITHMETIC: EXAMPLES ONLY.

By the Rev. T. MITCHESON, B.A., Assistant-Master in the City of London School.

Part I., Crown 8vo, cloth, pp. 119, 1s. (Answers, 1s.; Teachers' Copy, with Answers, 1s. 6d.)

Part II., Crown 8vo, cloth, pp. 200, 1s. 6d. (Answers, 1s.; Teachers' Copy, with Answers, 2s.)

Complete in One Volume, crown 8vo, cloth, 2s. (Answers, 1s. 6d.; Teachers' Copy, with Answers, 3s.)

By the same Author. New and enlarged Edition.

EXAMPLES IN ALGEBRA.

Crown 8vo, cloth, Pp. 96, 1s. with Answers, 1s. 6d. Answers only, 1s.

3 2044 102 915 568